

SUPPLEMENT TO “TESTING FOR STOCHASTIC MONOTONICITY”  
(*Econometrica*, Vol. 77, No. 2, March 2009, 585–602)

BY SOKBAE LEE, OLIVER LINTON, AND YOON-JAE WHANG

MATHEMATICAL APPENDIX: PROOFS

A.1. *Informal Discussion of the Proof Technique*

ALTHOUGH THE TEST IS EASY to implement, proving Theorem 3.1 involves several lengthy steps. Since establishing these steps requires techniques that are not commonly used in econometrics, we now give an informal description of our proof techniques and provide some discussions behind them. Specifically, our proof of Theorem 3.1 consists of the following three steps:

Step 1. The asymptotic approximation of  $\widehat{U}_n(y, x)/c_n(x)$  by a Gaussian process (Appendix A.2).

Step 2. The asymptotic approximation of the excursion probability of the maximum of the Gaussian process on a fixed set (Appendix A.3).

Step 3. The asymptotic approximation of the excursion probability of the maximum of the Gaussian process on an increasing set (Appendix A.4).

In particular, in Step 1, we show that  $\widehat{U}_n(y, x)/c_n(x)$  can be approximated uniformly over  $(y, x)$  by  $\xi_n[F_Y(y), h_n^{-1}x]$ , where  $F_Y(\cdot)$  is the cumulative distribution function (c.d.f.) of  $Y$  and  $\xi_n$  is a sequence of Gaussian processes  $\{\xi_n(u, s) : (u, s) \in [0, 1] \times [0, h_n^{-1}]\}$  with continuous sample paths such that

$$(A1) \quad E[\xi_n(u, s)] = 0, \\ E[\xi_n(u_1, s_1)\xi_n(u_2, s_2)] = [\min(u_1, u_2) - u_1u_2]\rho(s_1 - s_2)$$

for  $u, u_1, u_2 \in [0, 1]$  and  $s, s_1, s_2 \in [0, h_n^{-1}]$ , where  $\rho(\cdot)$  is some known smooth function. See Appendix A.2 for the exact form of  $\rho(\cdot)$ .

First of all, note that by Step 1, taking the supremum of  $\widehat{U}_n(y, x)/c_n(x)$  over  $(y, x)$  corresponds to taking the supremum of  $\xi_n[F_Y(y), h_n^{-1}x]$  over  $(y, x)$  asymptotically. Since  $F_Y$  is the c.d.f. and  $h_n \rightarrow 0$ , this means that we need to take the supremum of the Gaussian process  $\xi_n$  over the product space of a fixed set (in the direction of  $y$ ) and an increasing set (in the direction of  $x$ ).

In general, it is expected that the asymptotic distribution of a suitably normalized version of the supremum of a Gaussian process over an increasing set converges to one of extreme value distributions. If the supremum is taken over Gaussian processes with a one-dimensional parameter, then the corresponding probability theory and applications on statistical problems are well understood. See, for example, Leadbetter, Lindgren, and Rootzén (1983). However, for Gaussian processes with multidimensional parameters (often called Gaussian fields), the probability theory is less developed and applications on statistical problems are rare. Unfortunately, we need to deal with  $\xi_n(u, s)$  that has two

parameters and approximate the distribution of its supremum over an increasing set. These tasks are Steps 2 and 3. The important reference we have used to carry out Steps 2 and 3 is Piterbarg (1996), who developed a general theory for approximations of the suprema of Gaussian fields.

Once Step 2 is established, then there is a general approximation method to achieve Step 3. Thus, Step 2 is the critical step in proving Theorem 3.1. Note that the covariance function of  $\xi_n$  in (A1) is the product of a Brownian bridge covariance function and a stationary covariance function. In this paper, we develop a new result for the excursion probability of the maximum of the Gaussian process  $\xi_n$  (Theorem A.2). To be specific, the approximating Gaussian process contains both a stationary and a nonstationary part, and therefore we need to extend existing results that only apply to either one or the other case. For example, see Section 7 of Piterbarg (1996) for the stationary case and Sections 8 and 9 of Piterbarg (1996) for the nonstationary case, but to our best knowledge, there is no known result regarding our case in the literature.

### A.2. Gaussian Process Approximation

Let  $f_X(\cdot)$ ,  $F_X(\cdot)$ , and  $F_Y(\cdot)$ , respectively, denote the probability density function (p.d.f.), the c.d.f. of  $X$ , and the c.d.f. of  $Y$ . Define

$$\rho(s) = \frac{\int q(z)q(z-s)K(z)K(z-s)dz}{\int q^2(z)K^2(z)dz},$$

where  $q(u) = \int \text{sgn}(u-w)K(w)dw$  was defined in the main text. Let  $\xi(u, s)$  denote a Gaussian process  $\{\xi(u, s) : (u, s) \in [0, 1] \times \mathbb{R}\}$  with continuous sample paths such that

$$\begin{aligned} E[\xi(u, s)] &= 0, \\ E[\xi(u_1, s_1)\xi(u_2, s_2)] &= [\min(u_1, u_2) - u_1u_2]\rho(s_1 - s_2) \end{aligned}$$

for  $u, u_1, u_2 \in [0, 1]$  and  $s, s_1, s_2 \in \mathbb{R}$ . Define  $\mathcal{X}_n = [0, 1/h_n]$  and let  $\xi_n$  be the restriction of  $\xi$  to  $[0, 1] \times \mathcal{X}_n$ .

**THEOREM A.1:** *Let Assumption 3.1 hold. Let  $h_n$  satisfy*

$$h_n(\log n)^{1/2} \rightarrow 0, \quad nh_n^3 \rightarrow \infty, \quad \text{and} \quad nh_n^2/(\log n)^2 \rightarrow \infty.$$

*Then there exists a sequence of Gaussian processes  $\{\xi_n(u, s) : (u, s) \in [0, 1] \times \mathcal{X}_n\}$  with continuous sample paths such that*

$$\begin{aligned} E[\xi_n(u, s)] &= 0, \\ E[\xi_n(u_1, s_1)\xi_n(u_2, s_2)] &= [\min(u_1, u_2) - u_1u_2]\rho(s_1 - s_2) \end{aligned}$$

for  $u, u_1, u_2 \in [0, 1]$  and  $s, s_1, s_2 \in \mathcal{X}_n$ , and that

$$\begin{aligned} & \sup_{(y,x) \in \mathcal{Y} \times \mathcal{X}} \left| n^{1/2} \frac{\widehat{U}_n(y, x)}{\widehat{\sigma}_n(x)} - \xi_n[F_Y(y), h_n^{-1}x] \right| \\ &= O_p(n^{-1/2}h_n^{-3/2} + n^{-1/4}h_n^{-1/2}(\log n)^{1/2} + h_n(\log h_n^{-1})^{1/2}). \end{aligned}$$

PROOF: The proof of the theorem follows closely Theorem 3.1 of Ghosal, Sen, and van der Vaart (2000). In particular, the theorem can be proved by combining arguments almost identical to those used in the proof of Theorem 3.1 of Ghosal, Sen, and van der Vaart (2000) with the lemmas proved in Section A.6. The only difference here is that because of the estimated  $X_i$ 's, an additional term of order  $O_p(n^{-1/2}h_n^{-3/2})$  appears. *Q.E.D.*

### A.3. Asymptotic Behavior of the Excursion Probability on the Fixed Set

Since the distribution of  $\xi_n(u, s)$  does not depend on  $n$ , for the purpose of deriving the distribution of the supremum statistic  $S_n$ , it suffices to consider the asymptotic behavior of the excursion probability of the maximum of the Gaussian process  $\xi(u, s)$  that has the same covariance function as  $\xi_n(u, s)$ .

We first consider the asymptotic behavior of the tail probability of the maximum of  $\xi(u, s)$  on a fixed set  $[0, 1] \times I$ , where  $I \equiv [0, L]$  is an interval with a fixed length  $L$ . Define

$$\Psi(a) = \frac{1}{\sqrt{2\pi}} \int_a^\infty \exp\left(-\frac{1}{2}x^2\right) dx.$$

THEOREM A.2: *Let  $\lambda$  denote the quantity defined in Theorem 3.1. In addition, let  $I = [0, L]$ . Then*

$$\Pr\left(\max_{(u,s) \in [0,1] \times I} \xi(u, s) > a\right) = L \left(\frac{8\lambda}{\pi}\right)^{1/2} a \exp(-2a^2)[1 + o(1)]$$

as  $a \rightarrow \infty$ .

The following lemmas are useful to prove Theorem A.2.

LEMMA A.1: *Let  $\Pi_\delta = [1/2 - \delta(a), 1/2 + \delta(a)]$ , where  $\delta(a) = a^{-1} \log a$ . Then*

$$\Pr\left(\max_{(u,s) \in [0,1] \times I} \xi(u, s) > a\right) = \Pr\left(\max_{(u,s) \in \Pi_\delta \times I} \xi(u, s) > a\right)[1 + o(1)]$$

as  $a \rightarrow \infty$ .

PROOF: For all sufficiently large  $a$ ,

$$\begin{aligned}
 \text{(A2)} \quad & \Pr\left(\max_{(u,s) \in \Pi_\delta \times I} \xi(u, s) > a\right) \\
 & \leq \Pr\left(\max_{(u,s) \in [0,1] \times I} \xi(u, s) > a\right) \\
 & \leq \Pr\left(\max_{(u,s) \in \Pi_\delta \times I} \xi(u, s) > a\right) + \Pr\left(\max_{(u,s) \in \{[0,1] \setminus \Pi_\delta\} \times I} \xi(u, s) > a\right).
 \end{aligned}$$

Note that

$$\begin{aligned}
 & E[\xi(u_1, s_1) - \xi(u_2, s_2)]^2 \\
 & = u_1(1 - u_1) + u_2(1 - u_2) - 2[\min(u_1, u_2) - u_1 u_2] \rho(s_1 - s_2).
 \end{aligned}$$

Furthermore, by some straightforward manipulation,

$$E[\xi(u_1, s_1) - \xi(u_2, s_2)]^2 \leq C|u_1 - u_2| + |s_1 - s_2|$$

for some constant  $C$ . Thus, Assumption of Piterbarg (1996, p. 118) is satisfied. Then since

$$\max_{(u,s) \in \{[0,1] \setminus \Pi_\delta\} \times I} \sigma^2(u, s) \leq 1/4 - \delta(a)^2,$$

by Theorem 8.1 of Piterbarg (1996, p. 119), there exists a constant  $C$  such that

$$\begin{aligned}
 \text{(A3)} \quad & \Pr\left(\max_{(u,s) \in \{[0,1] \setminus \Pi_\delta\} \times I} \xi(u, s) > a\right) \\
 & \leq C \text{mes}(\{[0,1] \setminus \Pi_\delta\} \times I) a^4 \Psi\left(\frac{a}{[1/4 - \delta(a)^2]^{1/2}}\right),
 \end{aligned}$$

$\text{mes}(A)$  denotes the Lebesgue measure of a set  $A$ . Note that by (D.8) of Piterbarg (1996, p. 15), as  $a \rightarrow \infty$ ,

$$a^4 \Psi\left(\frac{a}{[1/4 - \delta(a)^2]^{1/2}}\right) \sim \frac{1}{\sqrt{2\pi}} a^3 \exp\left(\frac{-a^2/2}{1/4 - \delta(a)^2}\right),$$

where  $A \sim B$  stands for  $A/B \rightarrow 1$ . Also, for some fixed interior point  $\bar{s} \in I$ , we have

$$\Pr\left(\xi\left(\frac{1}{2}, \bar{s}\right) > a\right) = \Psi\left(\frac{a}{2}\right) \sim \frac{2}{\sqrt{2\pi}} a^{-1} \exp\left(\frac{-a^2/2}{4}\right).$$

Then it is easy to show that as  $a \rightarrow \infty$ , the probability on the left-hand side of (A3) converges to zero at a rate of  $\exp[-2a^2 + O(\log a)]$  and  $\Pr(\xi(1/2, \bar{s}) > a)$

converges to zero at a rate of  $\exp[-a^2/8 - O(\log a)]$ . Thus, the probability on the left-hand side of (A3) converges to zero faster than  $\Pr(\xi(1/2, \bar{s}) > a)$ . Since  $\Pr(\xi(1/2, \bar{s}) > a) \leq \Pr(\max_{(u,s) \in \Pi_\delta \times I} \xi(u, s) > a)$ ,

$$\Pr\left(\max_{(u,s) \in \{[0,1] \setminus \Pi_\delta\} \times I} \xi(u, s) > a\right) = o\left[\Pr\left(\max_{(u,s) \in \Pi_\delta \times I} \xi(u, s) > a\right)\right].$$

Then the lemma follows immediately from (A2). *Q.E.D.*

Let  $\sigma^2(u, s) = u(1-u)$  and  $r[(u_1, s_1), r(u_2, s_2)] = [\min(u_1, u_2) - u_1 u_2] \rho(s_1 - s_2)$ , respectively, denote the variance and covariance functions of  $\xi(u, s)$ .

LEMMA A.2: As  $u \rightarrow 1/2$ ,

$$(A4) \quad \sigma^2(u, s) = \frac{1}{4} - \left(u - \frac{1}{2}\right)^2 [1 + o(1)].$$

Furthermore, as  $(u_1, u_2) \rightarrow (1/2, 1/2)$  and  $|s_1 - s_2| \rightarrow 0$ ,

$$(A5) \quad r[(u_1, s_1), r(u_2, s_2)] \\ = \frac{1}{4} - \frac{1}{2}|u_1 - u_2|[1 + o(1)] - \frac{\lambda}{8}(s_1 - s_2)^2[1 + o(1)] \\ - \frac{1}{2}\left(u_1 - \frac{1}{2}\right)^2 [1 + o(1)] - \frac{1}{2}\left(u_2 - \frac{1}{2}\right)^2 [1 + o(1)].$$

PROOF: The first result (A4) follows easily from a second-order Taylor series expansion of the variance of  $\xi(u, s)$  with respect to  $u$ . We now consider the second result (A5). In view of the proof of Theorem 9.2 of Piterbarg (1996, p. 138), note that as  $(u_1, u_2) \rightarrow (1/2, 1/2)$ ,

$$(A6) \quad \frac{\min(u_1, u_2) - u_1 u_2}{\sqrt{u_1(1-u_1)u_2(1-u_2)}} = 1 - \frac{1}{2} \frac{|u_1 - u_2|}{\sqrt{u_1(1-u_1)u_2(1-u_2)}} + o(|u_1 - u_2|).$$

Note that by (4.9) of Ghosal, Sen, and van der Vaart (2000),

$$(A7) \quad \rho(s_1 - s_2) = 1 - \frac{\lambda(s_1 - s_2)^2}{2} + o(|s_1 - s_2|^2)$$

as  $|s_1 - s_2| \rightarrow 0$ . As in (A4), a Taylor series expansion of  $\sigma(u, s)$  around  $u = 1/2$  gives

$$\sigma(u, s) = \frac{1}{2} - \left(u - \frac{1}{2}\right)^2 [1 + o(1)] \quad \text{as } u \rightarrow \frac{1}{2}$$

for any  $s \in I$ . Thus, we have

$$(A8) \quad \begin{aligned} & \sqrt{u_1(1-u_1)u_2(1-u_2)} \\ &= \frac{1}{4} - \frac{1}{2} \left(u_1 - \frac{1}{2}\right)^2 [1 + o(1)] - \frac{1}{2} \left(u_2 - \frac{1}{2}\right)^2 [1 + o(1)] \end{aligned}$$

as  $(u_1, u_2) \rightarrow (1/2, 1/2)$ . Then the lemma follows from combining (A6) and (A7) with (A8). *Q.E.D.*

Let  $\varepsilon > 0$  be a fixed constant. Define Gaussian processes  $\psi_1^-(u)$  and  $\psi_1^+(u)$  such that

$$\begin{aligned} \psi_1^-(u) &= \frac{\zeta_1^-(u)}{2^{3/2}[1 + 4(1 + \varepsilon)(u - 0.5)^2]} \quad \text{and} \\ \psi_1^+(u) &= \frac{\zeta_1^+(u)}{2^{3/2}[1 + 4(1 - \varepsilon)(u - 0.5)^2]}, \end{aligned}$$

where  $\zeta_1^-(u)$  and  $\zeta_1^+(u)$  are Gaussian stationary processes with zero means and the covariance functions  $r_1^-(u) = \exp[-4(1 - \varepsilon)|u|]$  and  $r_1^+(u) = \exp[-4(1 + \varepsilon)|u|]$ . In addition, define mean-zero stationary Gaussian processes  $\psi_2^-(s)$  and  $\psi_2^+(s)$  such that they are independent of  $\psi_1^-(u)$  and  $\psi_1^+(u)$ , and have the covariance functions of the form

$$\begin{aligned} r_2^-(s) &= \frac{1}{8}[1 - \lambda(1 - \varepsilon)s^2 + o(s^2)], \\ r_2^+(s) &= \frac{1}{8}[1 - \lambda(1 + \varepsilon)s^2 + o(s^2)], \end{aligned}$$

respectively. Finally, define

$$\psi^-(u, s) = \psi_1^-(u) + \psi_2^-(s) \quad \text{and} \quad \psi^+(u, s) = \psi_1^+(u) + \psi_2^+(s).$$

LEMMA A.3: *Let  $\varepsilon > 0$  be any fixed, arbitrarily small constant. Then for all sufficiently large  $a$ ,*

$$\begin{aligned} \Pr\left(\max_{(u,s) \in \Pi_\delta \times I} \psi^-(u, s) > a\right) &\leq \Pr\left(\max_{(u,s) \in \Pi_\delta \times I} \xi(u, s) > a\right) \\ &\leq \Pr\left(\max_{(u,s) \in \Pi_\delta \times I} \psi^+(u, s) > a\right). \end{aligned}$$

PROOF: As noted in the proofs of Theorems D.4 and 8.2 of Piterbarg (1996, pp. 23 and 133), the lemma follows from Lemma A.2 and the fact that the distribution of the maximum is monotone with respect to the variance and the Slepian inequality (see, for example, Theorem C.1 of Piterbarg (1996, p. 6)). *Q.E.D.*

LEMMA A.4: *Let  $\varepsilon > 0$  be any fixed, arbitrarily small constant. As  $a \rightarrow \infty$ ,*

$$(A9) \quad \Pr\left(\max_{u \in I_\delta} 2^{3/2} \psi_1^-(u) > a\right) = 2^{1/2} \frac{(1 - \varepsilon)}{(1 + \varepsilon)^{1/2}} \exp\left(\frac{-a^2}{2}\right) [1 + o(1)],$$

$$(A10) \quad \Pr\left(\max_{u \in I_\delta} 2^{3/2} \psi_1^+(u) > a\right) = 2^{1/2} \frac{(1 + \varepsilon)}{(1 - \varepsilon)^{1/2}} \exp\left(\frac{-a^2}{2}\right) [1 + o(1)].$$

PROOF: This lemma can be proved by one of results given in the proof of Theorem D.4 of Piterbarg (1996, p. 21). In particular, using the notation used in the proof of Theorem D.4 of Piterbarg (1996), the excursion probability of  $2^{3/2} \psi_1^-(u)$  can be obtained by the result of Case 1 with  $\alpha = 1$ ,  $\beta = 2$ ,  $b = 4(1 + \varepsilon)$ , and  $d = 4(1 - \varepsilon)$ . It follows from the second display on page 22 of Piterbarg (1996) that as  $a \rightarrow \infty$ ,

$$\Pr\left(\max_{u \in I_\delta} 2^{3/2} \psi_1^-(u) > a\right) = \frac{H_1 \Gamma(1/2) [4(1 - \varepsilon)]}{[4(1 + \varepsilon)]^{1/2}} a \Psi(a) [1 + o(1)],$$

where  $H_1$  is the Pickands constant with  $\alpha = 1$  (defined on pages 13 and 16 of Piterbarg (1996)) and  $\Gamma(\cdot)$  is the Gamma function. Note that  $\Gamma(1/2) = \sqrt{\pi}$ . Furthermore, by (9.6) of Piterbarg (1996, p. 138),  $H_1 = 1$  and by (D.8) of Piterbarg (1996, p. 15),

$$a \Psi(a) \sim (2\pi)^{-1/2} \exp(-a^2/2)$$

as  $a \rightarrow \infty$ . Therefore, (A9) follows immediately. The excursion probability of  $2^{3/2} \psi_1^+(u)$  can be obtained analogously. *Q.E.D.*

LEMMA A.5: *Let  $\varepsilon > 0$  be any fixed, arbitrarily small constant. As  $a \rightarrow \infty$ ,*

$$(A11) \quad \Pr\left(\max_{s \in I} 2^{3/2} \psi_2^-(s) > a\right) = \frac{[(\lambda/2)(1 - \varepsilon)]^{1/2} L}{\pi} \exp\left(\frac{-a^2}{2}\right) [1 + o(1)],$$

$$(A12) \quad \Pr\left(\max_{s \in I} 2^{3/2} \psi_2^+(s) > a\right) = \frac{[(\lambda/2)(1 + \varepsilon)]^{1/2} L}{\pi} \exp\left(\frac{-a^2}{2}\right) [1 + o(1)].$$

PROOF: Recall that  $I = [0, L]$ . By Theorem D.2 of Piterbarg (1996, p. 16) and a simple scaling of  $\psi_2^-(u)$ ,

$$\Pr\left(\max_{s \in [0, L]} 2^{3/2} \psi_2^-(s) > a\right) = H_2 L^* a \Psi(a) [1 + o(1)],$$

where  $H_2$  is the Pickands constant with  $\alpha = 2$  and  $L^* = [\lambda(1 - \varepsilon)]^{1/2} L$ . By (F.4) of Piterbarg (1996, p. 31),  $H_2 = 1/\sqrt{\pi}$ . Then (A11) follows im-

mediately. The excursion probability of  $2^{3/2}\psi_2^+(u)$  can be obtained similarly. *Q.E.D.*

**PROOF OF THEOREM A.2:** Let  $\varepsilon > 0$  be any fixed, arbitrarily small constant. Note that  $\psi^-(u, s)$  and  $\psi^+(u, s)$  are convolutions of  $\psi_1^-(u)$  and  $\psi_2^-(s)$  and of  $\psi_1^+(u)$  and  $\psi_2^+(s)$ , respectively. Then an application of Lemma 8.6 of Piterbarg (1996, p. 128) with Lemmas A.4 and A.5 gives

$$(A13) \quad \Pr\left(\max_{(u,s) \in H_\delta \times I} 2^{3/2}\psi^-(u, s) > a\right) \\ = L \frac{(1-\varepsilon)^{3/2}}{(1+\varepsilon)^{1/2}} \left(\frac{\lambda}{\pi}\right)^{1/2} a \exp\left(\frac{-a^2}{4}\right) [1 + o(1)],$$

$$(A14) \quad \Pr\left(\max_{(u,s) \in H_\delta \times I} 2^{3/2}\psi^+(u, s) > a\right) \\ = L \frac{(1+\varepsilon)^{3/2}}{(1-\varepsilon)^{1/2}} \left(\frac{\lambda}{\pi}\right)^{1/2} a \exp\left(\frac{-a^2}{4}\right) [1 + o(1)].$$

Then as  $a \rightarrow \infty$ , by Lemma A.1,

$$\Pr\left(\max_{(u,s) \in [0,1] \times I} 2^{3/2}\xi(u, s) > a\right) = L \left(\frac{\lambda}{\pi}\right)^{1/2} a \exp\left(\frac{-a^2}{4}\right) [1 + o(1)]$$

since the choice of  $\varepsilon$  can be made arbitrarily small and the constants on the right-hand sides of (A13) and (A14) are continuous at  $\varepsilon = 0$ . Therefore, the theorem follows immediately. *Q.E.D.*

#### A.4. Asymptotic Behavior of the Excursion Probability on the Increasing Set

**THEOREM A.3:** For any  $x$ ,

$$\Pr\left(4\beta_n \left\{ \max_{(u,s) \in [0,1] \times \mathcal{X}_n} \xi(u, s) - \beta_n \right\} < x\right) \\ = \exp\left\{-\exp\left(-x - \frac{x^2}{8\beta_n^2}\right) \left[1 + \frac{x}{4\beta_n^2}\right]\right\} + o(1),$$

where  $\beta_n$  is defined in equation (7) in the main text.

**PROOF:** This theorem can be proved using arguments similar to those used in the proof of Theorem G.1 of Piterbarg (1996). Note that the covariance function of  $\xi(u, s)$ , that is,  $r[(u_1, s_1), r(u_2, s_2)]$ , has compact support and in particular it is zero when  $|s_1 - s_2| > 2$ . Define an increasing sequence  $m_n$  such

that  $m_n \rightarrow \infty$  but  $m_n h_n \rightarrow 0$  as  $n \rightarrow \infty$ . That is,  $m_n$  converges to infinity slower than  $h_n^{-1}$ . Further, define sequences of sets

$$\begin{aligned} I_k &= [k(m_n h_n)^{-1}, (k+1)(m_n h_n)^{-1} - 2], \\ J_k &= [(k+1)(m_n h_n)^{-1} - 2, (k+1)(m_n h_n)^{-1}] \end{aligned}$$

for  $k = 0, 1, \dots, m_n - 1$ . Then we have

$$\begin{aligned} \text{(A15)} \quad & \Pr\left(\max_{(u,s) \in [0,1] \times \mathcal{X}_n} \xi(u, s) < a\right) \\ &= \Pr\left(\max_{(u,s) \in [0,1] \times [\cup_k I_k]} \xi(u, s) < a\right) \\ &\quad - \Pr\left(\max_{(u,s) \in [0,1] \times [\cup_k I_k]} \xi(u, s) < a, \max_{(u,s) \in [0,1] \times [\cup_k J_k]} \xi(u, s) \geq a\right). \end{aligned}$$

We first consider the first probability on the right-hand side of (A15). Let  $c^* = (\frac{8\lambda}{\pi})^{1/2}$ . For each  $x$ , choose  $a_n = \beta_n + x/(4\beta_n)$ , where  $\beta_n$  is the largest solution to the equation

$$\text{(A16)} \quad h_n^{-1} c^* \beta_n \exp(-2\beta_n^2) = 1.$$

Since  $I_k$ 's are separated by the diameter of the support and the distribution of  $\xi(u, s)$  is stationary in the direction of  $s$ , it follows from Theorem A.2 that

$$\begin{aligned} & \Pr\left(\max_{(u,s) \in [0,1] \times [\cup_k I_k]} \xi(u, s) < a_n\right) \\ &= \left[1 - \Pr\left(\max_{(u,s) \in [0,1] \times I_0} \xi(u, s) \geq a_n\right)\right]^{m_n} \\ &= \exp\left(m_n \log\left[1 - \Pr\left(\max_{(u,s) \in [0,1] \times I_0} \xi(u, s) \geq a_n\right)\right]\right) \\ &= \exp\left(-m_n \Pr\left(\max_{(u,s) \in [0,1] \times I_0} \xi(u, s) \geq a_n\right)\right) \\ &\quad + O\left(m_n \left[\Pr\left(\max_{(u,s) \in [0,1] \times I_0} \xi(u, s) \geq a_n\right)\right]^2\right) \\ &= \exp\{-m_n[(m_n h_n)^{-1} - 2]c^* a_n \exp(-2a_n^2)[1 + o(1)]\} + O(h_n), \end{aligned}$$

so that

$$\begin{aligned} \text{(A17)} \quad & \Pr\left(\max_{(u,s) \in [0,1] \times [\cup_k I_k]} \xi(u, s) < a_n\right) \\ &= \exp\left\{-\exp\left(-x - \frac{x^2}{8\beta_n^2}\right)\left[1 + \frac{x}{4\beta_n^2}\right]\right\} + o(1). \end{aligned}$$

Now consider the second probability on the right-hand side of (A15). Note that again using Theorem A.2 and the fact that the distribution of  $\xi(u, s)$  is stationary in the direction of  $s$ ,

$$\begin{aligned} & \Pr\left(\max_{(u,s) \in [0,1] \times [\cup_k I_k]} \xi(u, s) < a_n, \max_{(u,s) \in [0,1] \times [\cup_k J_k]} \xi(u, s) \geq a_n\right) \\ & \leq \Pr\left(\max_{(u,s) \in [0,1] \times [\cup_k J_k]} \xi(u, s) \geq a_n\right) \leq m_n \Pr\left(\max_{(u,s) \in [0,1] \times J_1} \xi(u, s) \geq a_n\right) \\ & = m_n \Pr\left(\max_{(u,s) \in [0,1] \times [0,2]} \xi(u, s) \geq a_n\right) = O(m_n h_n) = o(1). \end{aligned}$$

This and (A17) together prove the theorem. Q.E.D.

### A.5. Proofs of Theorems 3.1 and 3.2

PROOF OF THEOREM 3.1: Since  $\beta_n[n^{-1/2}h_n^{-3/2} + n^{-1/4}h_n^{-1/2}(\log n)^{1/2} + h_n(\log n)^{1/2}] \rightarrow 0$ , the main theorem 3.1 is an immediate consequence of Theorems A.1 and A.3. Q.E.D.

PROOF OF THEOREM 3.2: The theorem can be proved by arguments similar to those used to prove Theorem 5.1 of Ghosal, Sen, and van der Vaart (2000). In fact, when  $F_x(y|x) > 0$  for some  $(y, x)$ ,  $S_n$  is of order  $O_p(n^{1/2}h_n^{3/2})$  and the consistency follows from the restriction that  $nh_n^3/\log h_n^{-1} \rightarrow \infty$ . Q.E.D.

### A.6. Lemmas for Proving Theorem A.1

Define

$$\begin{aligned} & V_n(y, x, \theta) \\ & = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} [1(Y_i \leq y) - 1(Y_j \leq y)] \operatorname{sgn}[\psi(W_i, \theta) - \psi(W_j, \theta)] \\ & \quad \times K_{h_n}[\psi(W_i, \theta) - x] K_{h_n}[\psi(W_j, \theta) - x], \end{aligned}$$

so that  $\widehat{U}_n(y, x) = V_n(y, x, \widehat{\theta})$ . Also, since  $F_x(y|x) \equiv 0$ , define the projection of  $V_n(y, x, \theta)$  by

$$\begin{aligned} & \widehat{V}_n(y, x, \theta) \\ & = 2n^{-1} \sum_{i=1}^n [1(Y_i \leq y) - F(y)] \\ & \quad \times \int \operatorname{sgn}[\psi(W_i, \theta) - \psi(\tilde{w}, \theta)] K_{h_n}[\psi(\tilde{w}, \theta) - x] dF_W(\tilde{w}) \\ & \quad \times K_{h_n}[\psi(W_i, \theta) - x]. \end{aligned}$$

LEMMA A.6: *Let  $\Theta$  denote a neighborhood of  $\theta_0$ .*

$$\sup_{(y,x,\theta) \in \mathcal{Y} \times \mathcal{X} \times \Theta} |V_n(y, x, \theta) - \widehat{V}_n(y, x, \theta)| = O_p(n^{-1}h_n^{-2}).$$

PROOF: The proof is similar to that of Lemma 3.1 of Ghosal, Sen, and van der Vaart (2000). Hence, we will only indicate the differences. Consider a class of functions  $\mathcal{M} = \{m_{(y,x,\theta)} : (y, x, \theta) \in \mathcal{Y} \times \mathcal{X} \times \Theta\}$ , where

$$\begin{aligned} m_{(y,x,\theta)}((y_1, w_1), (y_2, w_2)) \\ &= [1(y_1 \leq y) - 1(y_2 \leq y)] \operatorname{sgn}[\psi(w_1, \theta) - \psi(w_2, \theta)] \\ &\quad \times K_{h_n}[\psi(w_1, \theta) - x] K_{h_n}[\psi(w_2, \theta) - x]. \end{aligned}$$

This class is contained in the product of the classes

$$\begin{aligned} \mathcal{M}_1 &= \{1(y_1 \leq y) - 1(y_2 \leq y) : y \in \mathcal{Y}\}, \\ \mathcal{M}_2 &= \left\{ K \left( \frac{\psi(w_1, \theta) - x}{h_n} \right) : (x, \theta) \in \mathcal{X} \times \Theta \right\}, \\ \mathcal{M}_3 &= \left\{ K \left( \frac{\psi(w_2, \theta) - x}{h_n} \right) : (x, \theta) \in \mathcal{X} \times \Theta \right\}, \\ \mathcal{M}_4 &= \{h_n^{-2} \operatorname{sgn}[\psi(w_1, \theta) - \psi(w_2, \theta)] \\ &\quad \times 1\{|\psi(w_1, \theta) - \psi(w_2, \theta)| \leq 2h_n\} : \theta \in \Theta\}. \end{aligned}$$

Since  $\theta$  is finite dimensional and  $K$  is of bounded variation,  $\mathcal{M}$  is a Vapnik–Červonenkis (VC) class with the envelope function  $Ch_n^{-2}$  with some positive finite constant  $C$ , by Lemmas 2.6.15 and 2.6.18 of van der Vaart and Wellner (1996). Then using Theorem 2.6.7 of van der Vaart and Wellner (1996) and following the proof of Lemma 3.1 of Ghosal, Sen, and van der Vaart (2000), we have, for some finite constant  $C$ ,

$$E \left[ \sup_{(y,x,\theta) \in \mathcal{Y} \times \mathcal{X} \times \Theta} |V_n(y, x, \theta) - \widehat{V}_n(y, x, \theta)| \right] \leq Cn^{-1}h_n^{-2},$$

which gives the conclusion of the lemma. Q.E.D.

LEMMA A.7:

$$\sup_{(y,x) \in \mathcal{Y} \times \mathcal{X}} |\widehat{U}_n(y, x) - \widehat{V}_n(y, x, \theta_0)| = O_p(n^{-1/2}).$$

PROOF: Note that by Assumption 3.1(h),

$$\begin{aligned}
& |\widehat{V}_n(y, x, \widehat{\theta}) - \widehat{V}_n(y, x, \theta_0)| \\
&= \left| 2n^{-1} \sum_{i=1}^n [1(Y_i \leq y) - F(y)] \right. \\
&\quad \times \left\{ \int \operatorname{sgn}[\psi(W_i, \widehat{\theta}) - \psi(\tilde{w}, \widehat{\theta})] K_{h_n}[\psi(\tilde{w}, \widehat{\theta}) - x] dF_W(\tilde{w}) \right. \\
&\quad \times K_{h_n}[\psi(W_i, \widehat{\theta}) - x] \\
&\quad \left. - \int \operatorname{sgn}[\psi(W_i, \theta_0) - \psi(\tilde{w}, \theta_0)] K_{h_n}[\psi(\tilde{w}, \theta_0) - x] dF_W(\tilde{w}) \right. \\
&\quad \left. \times K_{h_n}[\psi(W_i, \theta_0) - x] \right\} \Big| \\
&\leq C \left[ \left\| \widehat{\theta} - \theta_0 \right\| n^{-1} \sum_{i=1}^n K_{h_n}[\psi(W_i, \widehat{\theta}) - x] \right. \\
&\quad \left. + n^{-1} \sum_{i=1}^n \{K_{h_n}[\psi(W_i, \widehat{\theta}) - x] - K_{h_n}[\psi(W_i, \theta_0) - x]\} \right]
\end{aligned}$$

for some positive constant  $C < \infty$ , which is independent of  $(y, x)$ . Also, note that using the standard empirical process method (for example, van der Vaart and Wellner (1996)), it is straightforward to show that for a  $n^{-1/2}$  neighborhood  $\Theta_n$  of  $\theta_0$ ,

$$\begin{aligned}
& \sup_{(x, \theta) \in \mathcal{X} \times \Theta_n} n^{-1} \sum_{i=1}^n K_{h_n}[\psi(W_i, \theta) - x] = O_p(1), \\
& \sup_{(x, \theta) \in \mathcal{X} \times \Theta_n} n^{-1} \left| \sum_{i=1}^n \{K_{h_n}[\psi(W_i, \theta) - x] - K_{h_n}[\psi(W_i, \theta_0) - x]\} \right| \\
&= O_p(n^{-1/2}).
\end{aligned}$$

Then the lemma follows from the root- $n$  consistency of  $\widehat{\theta}$  and Lemma A.6 since  $\widehat{U}_n(y, x) = \widehat{V}_n(y, x, \widehat{\theta})$ . *Q.E.D.*

Define

$$\begin{aligned}
\phi_{n,y,x}(Y, X) &= 2[1(Y \leq y) - F_Y(y)] \\
&\quad \times \int \operatorname{sgn}(X - \tilde{x}) K_{h_n}(\tilde{x} - x) dF_X(\tilde{x}) K_{h_n}(X - x).
\end{aligned}$$

LEMMA A.8: *There exists a sequence of Gaussian processes  $G_n(\cdot)$ , indexed by  $\mathcal{Y} \times \mathcal{X}$ , with continuous sample paths and with*

$$\begin{aligned} E[G_n(y, x)] &= 0 \quad \text{for } (y, x) \in \mathcal{Y} \times \mathcal{X}, \\ E[G_n(y_1, x_1)G_n(y_2, x_2)] &= E[\phi_{n,y_1,x_1}(Y, X)\phi_{n,y_2,x_2}(Y, X)] \end{aligned}$$

for  $(y_1, x_1)$ , and  $(y_2, x_2) \in \mathcal{Y} \times \mathcal{X}$ , such that

$$\sup_{(y,x) \in \mathcal{Y} \times \mathcal{X}} |n^{1/2}\widehat{V}_n(y, x, \theta_0) - G_n(y, x)| = O(n^{-1/4}h_n^{-1}(\log n)^{1/2}) \quad a.s.$$

PROOF: As in the proof of Lemma 3.2 of Ghosal, Sen, and van der Vaart (2000), we use Theorem 1.1 of Rio (1994). Since it can be proved using arguments identical to those used to prove Lemma 3.2 of Ghosal, Sen, and van der Vaart (2000), we will only highlight the differences. To apply Rio's theorem, we rewrite  $\varphi_{n,y,x}(Y, X)$  as

$$\begin{aligned} \phi_{n,y,x}(Y, X) &= 2[1(U \leq u) - u] \\ &\quad \times \int \text{sgn}(X - \tilde{x})K_{h_n}(\tilde{x} - x) dF_X(\tilde{x})K_{h_n}(X - x) \\ &\equiv \varphi_{n,u,x}(U, X), \end{aligned}$$

where  $U = F_Y(Y)$  and  $u = F_Y(y)$ . Then  $U$  is uniformly distributed in  $[0, 1] \equiv \mathcal{U}$ . Thus, Theorem 1.1 of Rio (1994) can be applied to a normalized empirical process associated with  $\varphi_{n,u,x}(U, X)$ . First, we verify that the class of functions  $(v, t) \mapsto h_n\varphi_{n,u,x}(v, t)$ , indexed by  $(u, x) \in \mathcal{U} \times \mathcal{X}$ , is uniformly of bounded variation (UBV). By the definition of Rio (1994), it suffices to show that

$$\sup_{(u,x) \in \mathcal{U} \times \mathcal{X}} \sup_{g \in \mathcal{D}_2([0,1]^2)} \left( \int_{\mathbb{R}^2} h_n\varphi_{n,u,x}(v, t) \text{div } g(v, t) dv dt / \|g\|_\infty \right) < \infty,$$

where  $\mathcal{D}_2([0, 1]^2)$  denotes the space of  $C^\infty$  functions with values in  $\mathbb{R}^2$  and with compact support included in  $[0, 1]^2$ ,  $\text{div}$  denotes the divergence, and  $\|g\|_\infty = \sup_{(v,t) \in \mathbb{R}^2} \|g(v, t)\|$  with  $\|\cdot\|$  being the usual Euclidean norm. To do so, note that for any  $g(v, t) \equiv (g_v(v, t), g_t(v, t))$ ,

$$\begin{aligned} &\int_{\mathbb{R}^2} \varphi_{n,u,x}(v, t) \text{div } g(v, t) dv dt \\ &= \int_{\mathbb{R}^2} 2[1(v \leq u) - u] \int \text{sgn}(t - \tilde{x})K_{h_n}(\tilde{x} - x) dF_X(\tilde{x})K_{h_n}(t - x) \\ &\quad \times \left[ \frac{\partial g_v(v, t)}{\partial v} + \frac{\partial g_t(v, t)}{\partial t} \right] dv dt \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{R}} \int_{\mathbb{R}} 2[1(v \leq u) - u] \frac{\partial g_v(v, t)}{\partial v} dv \\
&\quad \times \int \operatorname{sgn}(t - \tilde{x}) K_{h_n}(\tilde{x} - x) dF_X(\tilde{x}) K_{h_n}(t - x) dt \\
&\quad + \int_{\mathbb{R}^2} 2[1(v \leq u) - u] \\
&\quad \times \int \operatorname{sgn}(t - \tilde{x}) K_{h_n}(\tilde{x} - x) dF_X(\tilde{x}) K_{h_n}(t - x) \frac{\partial g_t(v, t)}{\partial t} dv dt.
\end{aligned}$$

Then it is straightforward to verify that

$$\sup_{g \in \mathcal{D}_2([0, 1]^2)} \left( \int_{\mathbb{R}^2} \varphi_{n, u, x}(v, t) \operatorname{div} g(v, t) dv dt / \|g\|_{\infty} \right) = O(h_n^{-1})$$

uniformly over  $(u, x) \in \mathcal{U} \times \mathcal{X}$ . This implies that the class of functions  $\{h_n \varphi_{n, u, x} : (u, x) \in \mathcal{U} \times \mathcal{X}\}$  satisfies the UBV condition of Rio (1994). Furthermore, it is also straightforward to verify that

$$\sup_{g \in \mathcal{D}_2([a, b]^2)} \left( \int_{\mathbb{R}^2} \varphi_{n, u, x}(v, t) \operatorname{div} g(v, t) dv dt / \|g\|_{\infty} \right) = O(h_n^{-1}[b - a])$$

uniformly over  $(u, x) \in \mathcal{U} \times \mathcal{X}$ . This implies that the class of functions  $\{h_n \varphi_{n, u, x} : (u, x) \in \mathcal{U} \times \mathcal{X}\}$  also satisfies the local UBV (LUBV) condition of Rio (1994). We now verify that the class of functions  $\{h_n \varphi_{n, u, x} : (u, x) \in \mathcal{U} \times \mathcal{X}\}$  is a VC class. The function  $h_n \varphi_{n, u, x}$  is bounded by a constant uniformly in  $(u, z) \in \mathcal{U} \times \mathcal{X}$  and is obtained by taking an average of

$$2h_n[1(v \leq u) - 1(\tilde{u} \leq u)] \operatorname{sgn}(\tilde{x} - t) K_{h_n}(\tilde{x} - t) K_{h_n}(t - x)$$

over  $(\tilde{u}, \tilde{x})$ . Then it is easy to show that  $\{h_n \varphi_{n, u, x} : (u, x) \in \mathcal{U} \times \mathcal{X}\}$  is a VC class by using arguments similar to those used in the proof of Lemma 3.2 of Ghosal, Sen, and van der Vaart (2000), in particular equation (8.5). Finally, by applying Theorem 1.1 of Rio (1994), there exists a sequence of centered Gaussian processes  $G_n(u, x)$  with covariance

$$E[G_n(u_1, x_1)G_n(u_2, x_2)] = E[\varphi_{n, u_1, x_1}(U, X)\varphi_{n, u_2, x_2}(U, X)].$$

By switching back to the original variable  $Y$  and its corresponding index  $y$ , we obtain the desired result. *Q.E.D.*

Define

$$\sigma_n^2(x) = 4 \int \left[ \int \operatorname{sgn}(\bar{x} - \tilde{x}) K_{h_n}(\tilde{x} - x) dF_X(\tilde{x}) K_{h_n}(\bar{x} - x) \right]^2 dF_X(\bar{x})$$

and

$$\sigma^2(x) = 4 \left[ \int q^2(u) K^2(u) du \right] [f_X(x)]^3.$$

LEMMA A.9:

- (a)  $\sup_{x \in \mathcal{X}} |h_n \sigma_n^2(x) - \sigma^2(x)| = o(1),$
- (b)  $\liminf_{n \rightarrow \infty} h_n \inf_{x \in \mathcal{X}} \sigma_n^2(x) > 0,$
- (c)  $\sup_{x \in \mathcal{X}} |\hat{\sigma}_n^2(x) - \sigma_n^2(x)| = O_p(n^{-1/2} h_n^{-2}).$

PROOF: Parts (a) and (b) of the lemma follow directly from Lemma 3.3(a) and (b) of Ghosal, Sen, and van der Vaart (2000). To prove part (c) of the lemma, note that  $\hat{\sigma}_n^2(x)$  depends on the estimated  $X_i$ . To deal with this, let  $\tilde{\sigma}_n^2(x, \theta)$  be the same as  $\hat{\sigma}_n^2(x)$  except that  $\hat{X}_i$  is replaced by  $\psi(W_i, \theta)$ . As in the proof of Lemma A.6, modifying the proof of Lemma 3.3 of Ghosal, Sen, and van der Vaart (2000) gives

$$\sup_{x \in \mathcal{X}} \sup_{\theta \in \Theta} |\tilde{\sigma}_n^2(x, \theta) - E \tilde{\sigma}_n^2(x, \theta)| = O_p(n^{-1/2} h_n^{-2} + n^{-1} h_n^{-3} + n^{-3/2} h_n^{-4}),$$

where  $\Theta$  is a neighborhood of  $\theta_0$ . Then part (c) follows from the restriction on  $h_n$  and the fact that  $E \tilde{\sigma}_n^2(x, \theta)$  is Lipschitz continuous with respect to  $\theta$ . *Q.E.D.*

LEMMA A.10: *For the sequence of Gaussian processes  $\{G_n(y, x) : (y, x) \in \mathcal{Y} \times \mathcal{X}\}$  obtained in Lemma A.8, there corresponds a sequence of Gaussian processes  $\{\xi_n(u, s) : (u, s) \in [0, 1] \times \mathcal{X}_n\}$  with continuous sample paths such that*

$$\begin{aligned} E[\xi_n(u, s)] &= 0, \\ E[\xi_n(u_1, s_1) \xi_n(u_2, s_2)] &= [\min(u_1, u_2) - u_1 u_2] \rho(s_1 - s_2) \end{aligned}$$

for  $u, u_1, u_2 \in [0, 1]$  and  $s, s_1, s_2 \in \mathcal{X}_n$ , where

$$\sup_{(y, x) \in \mathcal{Y} \times \mathcal{X}} \left| \frac{G_n(y, x)}{\sigma_n(x)} - \xi_n[F_Y(y), h_n^{-1} x] \right| = O_p(h_n \sqrt{\log h_n^{-1}}).$$

PROOF: Let  $\mathcal{G}_n$  denote the class of functions  $\{g_{n,u,x} : (u, x) \in \mathcal{U} \times \mathcal{X}\}$ , where  $g_{n,u,x}(U, X) = \varphi_{n,u,x}(U, X) / \sigma_n(x)$ . Also, let  $\tilde{\mathcal{G}}_n$  denote the class of functions  $\{\tilde{g}_{n,u,x} : (u, x) \in \mathcal{U} \times \mathcal{X}\}$ , where

$$\begin{aligned} \tilde{g}_{n,u,x}(U, X) &= \tilde{\varphi}_{n,u,x}(U, X) / \tilde{\sigma}_{n,x}(X), \\ \tilde{\varphi}_{n,u,x}(U, X) &= [1(U \leq u) - u] \end{aligned}$$

$$\begin{aligned} & \times \int \operatorname{sgn}(X - \tilde{x}) K_{h_n}(\tilde{x} - x) d\tilde{x} K_{h_n}(X - x), \\ \tilde{\sigma}_{n,x}(X) &= \left[ \int \left( \int \operatorname{sgn}(\tilde{x} - \tilde{x}) K_{h_n}(\tilde{x} - x) d\tilde{x} \right)^2 [K_{h_n}(\tilde{x} - x)]^2 d\tilde{x} \right]^{1/2} \\ & \times [f_X(X)]^{1/2}. \end{aligned}$$

As explained in Remark 8.3 of Ghosal, Sen, and van der Vaart (2000), it is possible to extend Lemma A.8 in that there exists a sequence of Gaussian bridges, say  $\{B_n(g) : g \in \mathcal{G}_n \cup \tilde{\mathcal{G}}_n\}$ , with

$$E[B_n(g)] = 0, \quad E[B_n(g_1)B_n(g_2)] = \operatorname{cov}(g_1, g_2)$$

for all  $g, g_1, g_2 \in \mathcal{G}_n \cup \tilde{\mathcal{G}}_n$  and with continuous sample paths with respect to the  $L_2$  metric such that

$$G_n(u, x) = \sigma_n(x) B_n(\varphi_{n,u,x}),$$

where  $G_n(u, x)$  is defined in the proof of Lemma A.8. Now let  $\tilde{\xi}_n(u, x) = B_n(\tilde{g}_{n,u,x})$  and  $\gamma_n(u, x) = G_n(u, x)/\sigma_n(x) - \tilde{\xi}_n(u, x)$ . As in the proof of Lemma 3.4 of Ghosal, Sen, and van der Vaart (2000), note that  $\gamma_n(u, x)$  is a mean-zero Gaussian process with

$$E[\gamma_n(u_1, x_1)\gamma_n(u_2, x_2)] = E[(g_{n,u_1,x_1} - \tilde{g}_{n,u_1,x_1})(g_{n,u_2,x_2} - \tilde{g}_{n,u_2,x_2})].$$

Then the lemma can be proved using identical arguments to those used in the proof of Lemma 3.4 of Ghosal, Sen, and van der Vaart (2000). *Q.E.D.*

#### A.7. Proof of Theorem 6.1

This theorem can be proved using arguments similar to those used in the proof of Theorem 3.1. In particular, the following lemmas can be proved (the whose proofs are omitted here for brevity) and then the desired result follows.

Recall that

$$\rho(s) = \frac{\int q(z)q(z-s)K(z)K(z-s) dz}{\int q^2(z)K^2(z) dz},$$

where  $q(u) = \int \operatorname{sgn}(u-w)K(w) dw$ . Let  $\xi(u, \mathbf{s})$  denote a Gaussian process  $\{\xi(u, \mathbf{s}) : (u, \mathbf{s}) \in [0, 1] \times \mathbb{R}^d\}$  with continuous sample paths such that

$$E[\xi(u, \mathbf{s})] = 0,$$

$$E[\xi(u_1, \mathbf{s}_1)\xi(u_2, \mathbf{s}_2)] = [\min(u_1, u_2) - u_1 u_2] \prod_{j=1}^d \rho(s_{1j} - s_{2j})$$

for  $u, u_1, u_2 \in [0, 1]$  and  $\mathbf{s}, \mathbf{s}_1 \equiv (s_{11}, \dots, s_{1d}), \mathbf{s}_2 \equiv (s_{21}, \dots, s_{2d}) \in \mathbb{R}^d$ . Define  $\mathcal{X}_n = [0, 1/h_n]^d$  and let  $\xi_n$  be the restriction of  $\xi$  to  $[0, 1] \times \mathcal{X}_n$ .

LEMMA A.11: *Let Assumption 6.1 hold. Let  $h_n$  satisfy*

$$h_n(\log n)^{1/2} \rightarrow 0, \quad nh_n^{3d} \rightarrow \infty, \quad \text{and} \quad nh_n^{d+1}/(\log n)^{d+1} \rightarrow \infty.$$

*Then there exists a sequence of Gaussian processes  $\{\xi(u, \mathbf{s}) : (u, \mathbf{s}) \in [0, 1] \times \mathcal{X}_n\}$  with continuous sample paths such that*

$$E[\xi_n(u, \mathbf{s})] = 0,$$

$$E[\xi_n(u_1, \mathbf{s}_1)\xi_n(u_2, \mathbf{s}_2)] = [\min(u_1, u_2) - u_1u_2] \prod_{j=1}^d \rho(s_{1j} - s_{2j})$$

*for  $u, u_1, u_2 \in [0, 1]$  and  $\mathbf{s}, \mathbf{s}_1 \equiv (s_{11}, \dots, s_{1d}), \mathbf{s}_2 \equiv (s_{21}, \dots, s_{2d}) \in \mathcal{X}_n$ , and that*

$$\begin{aligned} & \sup_{(y, \mathbf{x}) \in \mathcal{Y} \times \mathcal{X}} \left| n^{1/2} \frac{\widehat{U}_n(y, \mathbf{x})}{\widehat{S}_n(x)} - \xi_n[F_Y(y), h_n^{-1}\mathbf{x}] \right| \\ & = O_p[n^{-1/2}h_n^{-3d/2} + n^{-1/2(d+1)}h_n^{-1/2}(\log n)^{1/2} + h_n(\log n)^{1/2}]. \end{aligned}$$

LEMMA A.12: *Let  $\lambda$  denote the quantity defined in Theorem 3.1 and let  $\mathbf{I} \equiv [0, L]^d$  be a rectangle with a fixed volume  $L^d$ . Then*

$$\Pr\left(\max_{(u, \mathbf{s}) \in [0, 1] \times \mathbf{I}} \xi(u, \mathbf{s}) > a\right) = L^d 2^{-(d-1)} \left(\frac{8\lambda}{\pi}\right)^{d/2} a^d \exp(-2a^2)[1 + o(1)]$$

*as  $a \rightarrow \infty$ .*

LEMMA A.13: *For any  $x$ ,*

$$\begin{aligned} & \Pr\left(4\mathbf{b}_n \left\{ \max_{(u, \mathbf{s}) \in [0, 1] \times \mathcal{X}_n} \xi(u, \mathbf{s}) - \mathbf{b}_n \right\} < x\right) \\ & = \exp\left\{-\exp\left(-x - \frac{x^2}{8\mathbf{b}_n^2}\right) \left[1 + \frac{x}{4\mathbf{b}_n^2}\right]^d\right\} + o(1), \end{aligned}$$

*where  $\mathbf{b}_n$  is defined in equation (16) in the main text.*

#### REFERENCES

- GHOSAL, S., A. SEN, AND A. W. VAN DER VAART (2000): "Testing Monotonicity of Regression," *Annals of Statistics*, 28, 1054–1082.
- LEADBETTER, M. R., G. LINDGREN, AND H. ROOTZÉN (1983): *Extremes and Related Properties of Random Sequences and Processes*. New York: Springer.

- PITERBARG, V. I. (1996): *Asymptotic Methods in the Theory of Gaussian Processes and Fields*, Translation of Mathematical Monographs, Vol. 148. Providence, RI: American Mathematical Society.
- RIO, E. (1994): "Local Invariance Principles and Their Application to Density Estimation," *Probability Theory and Related Fields*, 98, 21–45.
- VAN DER VAART, A. W., AND WELLNER, J. A. (1996): *Weak Convergence and Empirical Processes*. New York: Springer.

*Dept. of Economics, University College London, Gower Street, London WC1E 6BT, U.K.; [l.simon@ucl.ac.uk](mailto:l.simon@ucl.ac.uk),*

*Dept. of Economics, London School of Economics, Houghton Street, London WC2A 2AE, U.K.; [o.linton@lse.ac.uk](mailto:o.linton@lse.ac.uk), <http://econ.lse.ac.uk/staff/olinton/>,*  
*and*

*Dept. of Economics, Seoul National University, Seoul 151-742, Korea; [whang@snu.ac.kr](mailto:whang@snu.ac.kr), <http://plaza.snu.ac.kr/~whang>.*

*Manuscript received May, 2007; final revision received July, 2008.*