Testing for Stochastic Dominance Efficiency

Oliver Linton(London School of Economics)

Thierry Post(Erasmus University)

Yoon-Jae Whang(Seoul National University)

Nov. 2005

Cemmap Workshop

Outline

1. Introduction
2. Null Hypothesis
3. Test Statistics
4. Computational Strategy
5. Asymptotic Properties
6. Numerical Results
7. Conclusion

Motivation

- **Stochastic Dominance (SD)**:
  - A systematic rule to analyze the choice behavior of economic agents under uncertainty.
  - Less restrictions on the decision-maker preference and the statistical distributions of the choice alternatives than the traditional (mean-variance) approach

Motivation

- **Stochastic Dominance (SD)**:
  - A systematic rule to analyze the choice behavior of economic agents under uncertainty.
  - Less restrictions on the decision-maker preference and the statistical distributions of the choice alternatives than the traditional (mean-variance) approach
- **Stochastic Dominance Efficiency (SDE)**:

  Useful criterion as a portfolio screening/building device.

- **Existing Tests of SD Efficiency**:
  - Applicable to somewhat restrictive situations and/or
  - Lack of statistical power.

- **Existing Tests of SD Efficiency**:
  - Applicable to somewhat restrictive situations and/or
  - Lack of statistical power.
- **THE PURPOSE OF THIS PAPER**:

  To construct a new statistical test of SD efficiency that can be used under general settings and that might be more powerful than the existing tests.

Existing Results

- **Classical SD Tests (Finite Pairwise Comparisons)**

  - Anderson (1996), Davidson and Duclos (1997, 2000), McFadden (1989), Barrett and Donald (1999), Klecan, McFadden and McFadden (1991), Linton, Maasoumi and Whang (2004), etc.

  $\Rightarrow$ *Necessary but not Sufficient for SD efficiency*

- **SD Tests allowing for Diversification (Infinite Possible Comparisons)**

  - Post (2003), Post and Versijp (2004):

    - ⋆ *Based on a duality representation of the expected utility maximization.*

    - ⋆ *I.I.D., Equal means or Conservative test*

  - Kuosmanen (2004):

    - ⋆ *0-1 Mixed integer LP algorithm.*

Main Contributions

1. *An extension of LMW (2004) to the portfolio case (Infinite Comparisons).*
2. *Allow general cross-sectional and time series dependence.*
3. *Computional Strategy: New nested linear programming algorithm.*
4. *Standard statistical approach.*
   4.1 *Exact asymptotic critical values via subsampling and bootstrapping*
   4.2 *Investgation of power properties of the test*

Notation

- **Portfolio Returns**:
  - Y_t : *Benchmark asset (or portfolio)* with cdf F_Y
  - $X_t^\top \lambda$: Portfolio (with cdf F_λ) for $\lambda \in \Lambda_0$, where $X_t = (X_{1t}, \ldots, X_{Kt})^\top$,

    and $\Lambda_0 \subset \Lambda = \{\lambda \in \mathbb{R}^K : e^\top \lambda = 1\}$.

Notation

- **Portfolio Returns**:
  - Y_t : *Benchmark asset (or portfolio)* with cdf F_Y
  - $X_t^\top \lambda$: Portfolio (with cdf F_λ) for $\lambda \in \Lambda_0$, where $X_t = (X_{1t}, \ldots, X_{Kt})^\top$, and $\Lambda_0 \subset \Lambda = \{\lambda \in \mathbb{R}^K : e^\top \lambda = 1\}$.
- **Integrated CDF**: For a given integer $s \geq 1$,

$$G_\lambda^{(1)}(\cdot) = F_\lambda(\cdot); \; G_\lambda^{(s)}(x) = \int_{-\infty}^{x} G_\lambda^{(s-1)}(y)dy,$$

- **Definition (SD)**: $X_t^\top \lambda$ *s-th order (stochastically) dominates Y_t iff*

  (a) $G_\lambda^{(s)}(x) \leq G_Y^{(s)}(x)\ \forall x \in \mathcal{X}$, *with strict inequality for at least one x or*

  (b) $Eu(X_t^\top \lambda) \geq Eu(Y_t)\ \forall u \in \mathcal{U}_s$, where

  $\mathcal{U}_2 = \{u : u' \geq 0, u'' \leq 0\}, \mathcal{U}_3 = \{u \in \mathcal{U}_2 : u''' \geq 0\}$, etc.

- **Definition (SD)**: $X_t^\top \lambda$ *s-th order (stochastically) dominates* Y_t *iff*

  (a) $G_\lambda^{(s)}(x) \leq G_Y^{(s)}(x)$ $\forall x \in \mathcal{X}$, *with strict inequality for at least one* x *or*

  (b) $Eu(X_t^\top \lambda) \geq Eu(Y_t)$ $\forall u \in \mathcal{U}_s$, where

  $\mathcal{U}_2 = \{u : u' \geq 0, u'' \leq 0\}, \mathcal{U}_3 = \{u \in \mathcal{U}_2 : u''' \geq 0\}$, etc.

- Below, we shall denote $G_Y(x) = G_Y^{(s)}(x)$, $G_\lambda(x) = G_\lambda^{(s)}(x)$.

The Hypotheses

- **Definition (SSD Efficiency)**: Y_t *is (s-th order) SD efficient in* Λ_0 *iff there does not exist any portfolio in* $\{X_t^\top \lambda : \lambda \in \Lambda_0\}$ *that (s-th order) dominates it.*

The Hypotheses

- **Definition (SSD Efficiency)**: *Y_t is (s-th order) SD efficient in Λ_0 iff there does not exist any portfolio in $\{X_t^\top \lambda : \lambda \in \Lambda_0\}$ that (s-th order) dominates it.*

- **Hypotheses of interest:**

$$H_0 \quad : \quad Y_t \text{ is } (s\text{-th order) SD efficient in } \Lambda_0.$$

$$H_1 \quad : \quad \text{Negation of } H_0.$$

First Functional

- Consider

$$d_1 \quad = \quad \sup_{\lambda \in \Lambda_0} \inf_{x \in \mathcal{X}} [G_Y(x) - G_\lambda(x)]$$

where $\mathcal{X}$ denotes a compact set contained in the union of the supports of $G_Y(x)$ and $G_\lambda(x)$ for $\lambda \in \Lambda_0$.

- **Problem:**

$d_1 \leq 0$ *under* H_0, *but* $d_1 = 0$ *for some alternatives!!*

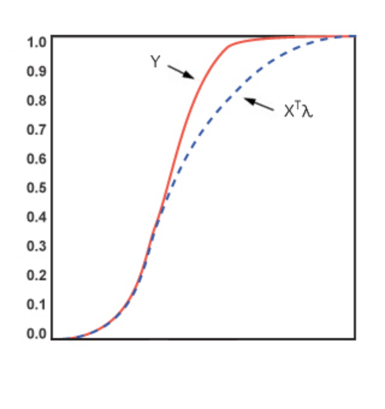


Figure 1. C.D.F's of Y and $X^\top\lambda$

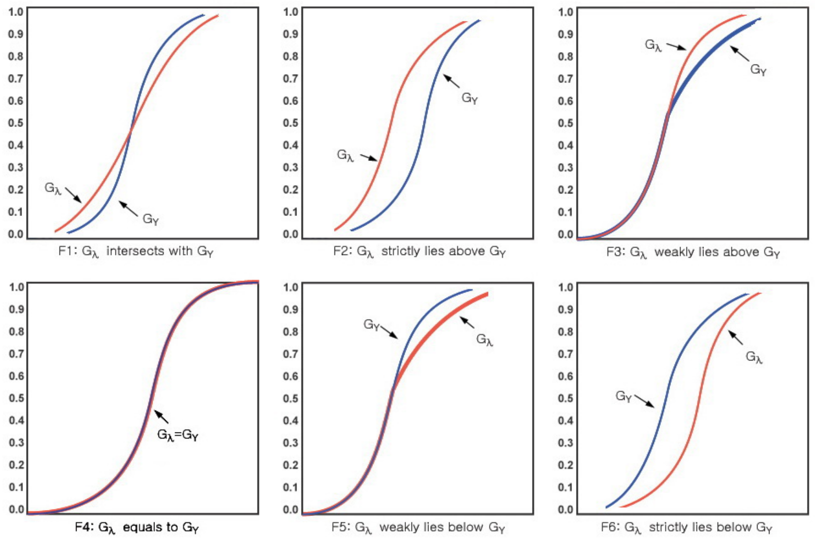
Relationships between (Integrated) CDFs

- Consider

$$H_0 : X_t^{\top}\lambda \ not \ SD \ Y_t \quad vs. \quad H_1 : X_t^{\top}\lambda \ \ SD \ Y_t$$

- **6 Possibilities:**

Case	Classification
1. G_λ intersects with G_Y	H_0
2. G_λ strictly lies above G_Y	H_0
3. G_λ weakly lies above G_Y	H_0
4. G_λ equals to G_Y	H_0
5. G_λ weakly lies below G_Y	H_1
6. G_λ strictly lies below G_Y	H_1


1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
Gλ
GY
F1: Gλ intersects with GY
F2: Gλ strictly lies above GY
F3: Gλ weakly lies above GY
Gλ=GY
F4: Gλ equals to GY
F5: Gλ weakly lies below GY
F6: Gλ strictly lies below GY

Our Functional

- **Decomposition of $\mathcal{X}$:**

$$\begin{aligned} A_\lambda^- &= \{x \in \mathcal{X} : G_Y(x) - G_\lambda(x) < 0\} \\ A_\lambda^= &= \{x \in \mathcal{X} : G_Y(x) - G_\lambda(x) = 0\} \\ A_\lambda^+ &= \{x \in \mathcal{X} : G_Y(x) - G_\lambda(x) > 0\}\,. \end{aligned}$$

- ϵ **- Neighborhood**:

$$(A_\lambda^=)^\epsilon = \{x + \eta \in \mathcal{X} : x \in A_\lambda^= \text{ and } |\eta| < \epsilon\}\,.$$

- **Our functional:**

$$d_*(\epsilon) = \sup_{\lambda \in \Lambda_0} \inf_{x \in B_\lambda^\varepsilon} [G_Y(x) - G_\lambda(x)]$$

where

$$B_\lambda^\epsilon = \begin{cases} \mathcal{X} \backslash (A_\lambda^=)^\epsilon & \text{if } A_\lambda^= \neq \mathcal{X} \\ \mathcal{X} & \text{otherwise} \end{cases}$$

  - Under $H_0, d_*(\epsilon) \leq 0$ for each $\epsilon \geq 0$, but under H_1, $d_*(\epsilon) > 0$ for some $\epsilon > 0$.
  - *Idea: Prevent the inner infimum ever being zero through equality on some part of* $\mathcal{X}$.

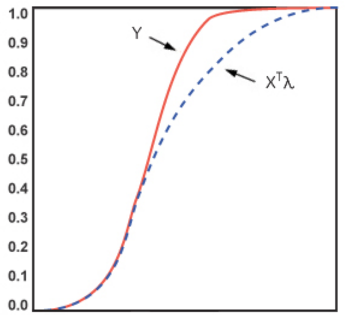


Figure 2. C.D.F's of Y and $X^\top \lambda$

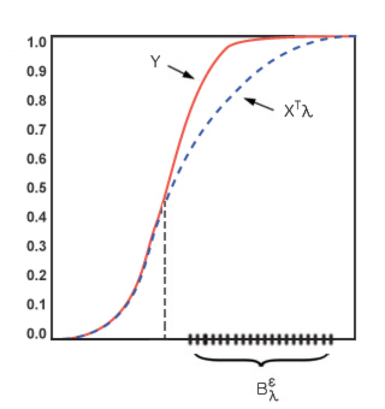


Figure 2. C.D.F's of Y and $X^\top\lambda$

The Test Statistic

- **Set Estimators:**

$$\begin{aligned}
\widehat{A}_\lambda^{=} &= \left\{x \in \mathcal{X} : \left|\widehat{G}_Y(x) - \widehat{G}_\lambda(x)\right| \leq k_T\right\}, \\
\left(\widehat{A}_\lambda^{=}\right)^{\epsilon_T} &= \left\{x + \eta \in \mathcal{X} :\ x \in \widehat{A}_\lambda^{=}, |\eta| < \epsilon_T\right\}, \\
\widehat{B}_\lambda^{\epsilon_T} &= \begin{cases} \mathcal{X} \backslash (\widehat{A}_\lambda^{=})^{\epsilon_T} & \text{if } \widehat{A}_\lambda^{=} \neq \mathcal{X} \\ \mathcal{X} & \text{otherwise} \end{cases}, \\
k_T &= c_0 \cdot (\log T/T)^{1/2}.
\end{aligned}$$

- **Test statistic**:

$$W_T = \sup_{\lambda \in \Lambda_0} \inf_{x \in \widehat{B}_\lambda^{\varepsilon_T}} Q_T(\lambda, x),$$

where

$$\begin{aligned} Q_T(\lambda, x) &= \sqrt{T}\left[\widehat{G}_Y(x) - \widehat{G}_\lambda(x)\right], \\ \widehat{G}_\lambda(x) &\equiv \widehat{G}_\lambda^{(s)}(x) = \int_{-\infty}^{x} \widehat{G}_\lambda^{(s-1)}(y)dy, \\ \widehat{G}_\lambda^{(1)}(x) &= \widehat{F}_{T\lambda}(x) = \frac{1}{T}\sum_{t=1}^{T} 1(X_t^\top \lambda \leq x). \end{aligned}$$

Computational Strategy

- **Objective Function:** (cf. Davidson and Duclos(2000))

$$
\begin{aligned}
& Q_T(\lambda, x) \\
= \quad & \frac{\sum_{t=1}^{T} \left\{ (x - Y_t)^{s-1} 1\,(Y_t \le x) - (x - X_t^\top \lambda)^{s-1} 1\left(X_t^\top \lambda \le x\right) \right\}}{(s-1)!\sqrt{T}}
\end{aligned}
$$

  - ⋆ Not smooth in λ (and/or x).
  - ⋆ Not easy to achieve the maximum with high accuracy when K is large.
  - ⋆ When $s \ge 2$, we can write the optimization problem as a one-dimensional grid search with embedded linear programming.

- **Profiling on the SSD Efficient Set**
  - Every SSD efficient portfolio is optimal for some $u \in \mathcal{U}_2$.
  - Russell and Seo (1989): Each $u \in \mathcal{U}_2$ can be represented by

$$u_\mu(x) = \min\{x - \mu, 0\} \text{ for } \mu \in \mathbb{R}$$

$\Rightarrow$ Every SSD efficient portfolio solves

$$\sup_{\lambda \in \Lambda_0} \frac{1}{T} \sum_{t=1}^{T} \min\{X_t^\top \lambda - \mu, 0\} \text{ for some } \mu.$$

- **LP programming Approach**:

$$\sup_{\theta \in \mathbb{R}^T, \lambda \in \mathbb{R}^K} \frac{1}{T} \sum_{t=1}^{T} \theta_t$$

$$\theta_t \leq \sum_{j=1}^{K} \lambda_j X_{jt} - \mu, \quad \theta_t \leq 0 \text{ for } t = 1, \ldots, T$$

$$\sum_{j=1}^{K} \lambda_j = 1, \quad \lambda_j \geq 0 \text{ for } j = 1, \ldots, K.$$

  - *Inner loop*: Compute $\lambda(\mu)$ for each $\mu \in M \subset \mathbb{R}$.
  - *Outer loop*: Compute the supremum over $\lambda(\in \Lambda_0) \subset \mathbb{R}^K$ by taking the maximum over $\lambda(\mu)$ for $\mu \in M \subset \mathbb{R}$ using a grid search.
  - Works for SD efficiency of order $s \geq 2$ due to the transitivity of SD rules.

Starting Values on the Mean-Variance Frontier

- **Standard Nelder Mead Algorithm**:
  - Very general but needs a good starting value especially in high-dimensional cases.
  - The MV efficient set is a natural starting point, because SD efficient set = MV efficient set for normal distributions.

- **MV efficient portfolio weight:** (cf. Campbell, Lo, McKinlay (1997))

Assuming $X_t \sim N(\mu, \Sigma)$,

$$\lambda_p = g + h\mu_p,$$

where

$$\begin{aligned} g &= \frac{1}{D}\left[B\Sigma^{-1}i - A\Sigma^{-1}\mu\right] \\ h &= \frac{1}{D}\left[C\Sigma^{-1}\mu - A\Sigma^{-1}i\right] \\ A &= i^{\top}\Sigma^{-1}\mu;\ C = i^{\top}\Sigma^{-1}i, \\ D &= BC - A^2, B = \mu^{\top}\Sigma^{-1}\mu. \end{aligned}$$

$\Rightarrow$ Take a grid of values of μ_p and obtain λ_p for this grid and then compute the test statistic. The optimal λ_p can be used as a starting value in some more general optimization algorithm.

Discussion: Why consider only $s \geq 2$?

- Risk aversion is a standard assumption in financial economics, e.g risk premium, portfolio diversification, and insurance contracts, etc.
- The definition of FSD efficiency in a portfolio context is ambiguous, see Post(2005).
- Our computational strategy breaks down for local risk seekers.
- The FSD criterion is too general to make an empirical test powerful in practice.

The Asymptotic Null Distribution

- Let

$$\nu_T(\lambda, x) = \sqrt{T}\left[\widehat{G}_Y(x) - \widehat{G}_\lambda(x) - G_Y(x) + G_\lambda(x)\right].$$

- Let $\widetilde{\nu}(\cdot,\cdot)$ be a mean zero Gaussian process on $\Lambda_0 \times \mathcal{X}$ with covariance function given by

$$C((\lambda_1, x_1), (\lambda_2, x_2)) = \lim_{T\to\infty} E\nu_T(\lambda_1, x_1)\nu_T(\lambda_2, x_2).$$

- **Theorem 1.** *Suppose Assumptions 1 and 2 hold. Then, under the null hypothesis, we have*

$$W_T \Rightarrow \Upsilon = \begin{cases} \sup_{\lambda \in \Lambda_0^=} \inf_{x \in \mathcal{X}} \left[\widetilde{\nu}(\lambda, x) \right] & \text{if } \Lambda_0^= \neq \varnothing \\ -\infty & \text{if } \Lambda_0^= = \varnothing \end{cases},$$

where

$$\Lambda_0^= = \{ \lambda \in \Lambda_0 : G_Y(x) = G_\lambda(x) \ \forall x \in \mathcal{X} \}.$$

Assumptions

- **Assumption 1.**

  (i) $\{(X_t^\top, Y_t)^\top : t = 1, \ldots, T\}$ *is a strictly stationary and* α- *mixing sequence with* $\alpha(m) = O(m^{-A})$ *for some* $A > (q-1)(1+q/2)$, *where* $X_t = (X_{1t}, \ldots, X_{Kt})^\top$ *and* q *is an even integer that satisfies* $q > 2(K+1)$.

  (ii) *The supports of* X_{kt} *and* Y_t *are compact* $\forall k = 1, \ldots, K$.

  (iii) *The distributions of* X_t *and* Y_t *are absolutely continuous with respect to Lebesgue measure and have bounded densities.*

- **Assumption 2.**

  (i) $\{\epsilon_T : T \geq 1\}$ *is a sequence of positive constants such that*

  $\lim_{T\to\infty} \epsilon_T = 0$ *and* $\epsilon_T > k_T \; \forall T \geq 1$.

  (ii) *For each* $x \in \mathcal{X}$, *constant* $C_1 > 0$ *and* $\lambda \in \Lambda_0$ *such that* $A_\lambda^= \neq \varnothing$, *we have:*

$$|G_Y(x) - G_\lambda(x)| \geq C_1 \min\left\{ \inf_{x' \in A_\lambda^=} \left| x - x' \right|, \epsilon_T \right\}$$

  for T *sufficiently large.*

Subsampling Critical Values

1. Calculate W_T using the full sample $\mathcal{W}_T = \{Z_t = (X_t^\top, Y_t)^\top : 1 \le t \le T\}$.
2. Generate subsamples $\mathcal{W}_{T,b,t} = \{Z_t, \dots, Z_{t+b-1}\}$ of size b, $1 \le t \le T - b + 1$.
3. Compute test statistics $W_{T,b,t}$ using $\mathcal{W}_{T,b,t}$.
4. Approximate the sampling distribution of W_T by
$$\widehat{S}_{T,b}(w) = \frac{1}{T-b+1} \sum_{t=1}^{T-b+1} 1\left(W_{T,b,t} \le w\right).$$
5. Get $s_{T,b}(1-\alpha) = \inf\{w : \widehat{S}_{T,b}(w) \ge \alpha\}$.
6. Reject H_0 if $W_T \; > \; s_{T,b}(1-\alpha)$.

An Example (Subsampling)

- Let $\{X_1, ..., X_T\}$ be iid with $N(\mu, 1)$. Consider $H_0 : \mu = 0$ vs. $H_1 : \mu > 0$.
- T-test: Let $W_T = \sqrt{T}\overline{X}_T$.

$$\text{Reject } H_0 \text{ if } \; W_T (= \sqrt{T}\overline{X}_T) > z(1-\alpha),$$

  where $z(1-\alpha) = (1-\alpha)$-th quantile of $N(0, 1)$.

- *Subsampling Test* **:** Assume $b \to \infty$ and $b/T \to 0$.

$$\text{Reject } H_0 \text{ if } \; W_T > s_{T,b}(1-\alpha),$$

  where $s_{T,b}(1-\alpha) = (1-\alpha)$-th quantile of subsampling distribution of $W_b = \sqrt{b}\overline{X}_b$.

- **Size**: Under H_0,

$$W_b = \sqrt{b}\overline{X}_b \Rightarrow N(0,1) \text{ as } b \to \infty.$$

Therefore,

$$\begin{aligned} & s_{T,b}(1-\alpha) \xrightarrow{p} z(1-\alpha) \\ & P\left(W_T > s_{T,b}(1-\alpha)\right) \\ \to \quad & P(N(0,1) > z(1-\alpha)) = \alpha. \end{aligned}$$

- **Consistency**: Under H_1,

$$W_b/\sqrt{b} \; = \overline{X}_b \xrightarrow{p} \mu > 0.$$

Therefore,

$$\begin{aligned} & P\left(W_T > s_{T,b}(1-\alpha)\right) \\ = \;& P\left(\sqrt{T/b}\,\overline{X}_T > s_{T,b}(1-\alpha)/\sqrt{b}\right) \\ = \;& P\left(\sqrt{T/b}\,\mu > \mu\right) + o(1) \to 1. \end{aligned}$$

- **Local Power**: Under $H_a : \mu = T^{-1/2}\delta$ for $\delta > 0$,

$$\begin{aligned} W_T &= \sqrt{T}\left(\overline{X}_T - \mu\right) + \delta \Rightarrow N(\delta, 1) \\ W_b &= \sqrt{b}(\overline{X}_b - \mu) + (b/T)^{1/2}\,\delta \Rightarrow N(0,1). \end{aligned}$$

Therefore,

$$\begin{aligned} & s_{T,b}(1-\alpha) \xrightarrow{p} z(1-\alpha) \\ & P\left(W_T > s_{T,b}(1-\alpha)\right) \\ \to\ & P\left(N(\delta,1) > z_{1-\alpha}\right) > \alpha. \end{aligned}$$

i.e., the same first-order non-trivial local power as the usual t-test and asymptotically locally unbiased.

Bootstrap Critical Values

1. Calculate W_T using the original full sample $\mathcal{W}_T$.
2. Generate bootstrap samples $\mathcal{W}_T^* = \{Z_t^* : t = 1, \ldots, T\}$ M -times.
3. Compute the recentred test statistic using $\mathcal{W}_T^*$:

   $W_T^* = \sup_{\lambda \in \Lambda_0} \inf_{x \in \mathcal{X}} Q_T^*(\lambda, x)$, where

   $$Q_T^*(\lambda, x) = \sqrt{T}\left[\widehat{G}_Y^*(x) - \widehat{G}_\lambda^*(x) - E^*\left(\widehat{G}_Y^*(x) - \widehat{G}_\lambda^*(x)\right)\right]$$

4. Approximate $\widehat{H}_T(w) = \frac{1}{M}\sum_{m=1}^{M} 1\left(W_{T,m}^* \leq w\right)$.
5. Get $h_T(1-\alpha) = \inf\{w : \widehat{H}_T(w) \geq \alpha\}$.
6. Reject H_0 if $W_T > h_T(1-\alpha)$.

Bootstrap vs. Subsampling?

- The recentring in bootstrap is crucial and is used to impose the least favorable case (LFC) of the null restriction, i.e.,

$$G_Y(x) = G_\lambda(x) \ \forall x \in \mathcal{X}, \ \forall \lambda \in \Lambda_0,$$

  which might be a special case of the boundary $(\Lambda_0^= \neq \emptyset)$ of the null hypothesis.

- If the LFC is true, the bootstrap test might be more efficient. Otherwise, it is not asymptotically similar on the boundary.

- The subsampling test is unbiased and asymptotically similar on the boundary.

Choice of Subsample Size

- **Assumption 3.**

  $P[l_T \leq \widehat{b}_T \leq u_T] \to 1$, *where* l_T *and* u_T *are integers satisfying*

  $1 \leq l_T \leq u_T \leq T,\ l_T \to \infty$ *and* $u_T/T \to 0$ *as* $T \to \infty$.

Asymptotic Size

- **Theorem 2 (Size).** *Suppose Assumptions 1-3 hold. Then, under the null hypothesis*, *we have*

$$\text{(a) } s_{T,\widehat{b}_T}(1-\alpha) \xrightarrow{p} \begin{cases} s(1-\alpha) & \text{if } \Lambda_0^{=} \neq \varnothing \\ -\infty & \text{if } \Lambda_0^{=} = \varnothing \end{cases}$$

$$\text{(b) } P[W_T > s_{T,\widehat{b}_T}(1-\alpha)] \to \begin{cases} \alpha & \text{if } \Lambda_0^{=} \neq \varnothing \\ 0 & \text{if } \Lambda_0^{=} = \varnothing \end{cases}$$

as $T \to \infty$, *where* $s(1-\alpha)$ *denotes the* $(1-\alpha)$*-th quantile of the asymptotic null distribution* $\sup_{\lambda \in \Lambda_0^{=}} \inf_{x \in \mathcal{X}} [\widetilde{\nu}(\lambda, x)]$ *of* W_T *given in Theorem 1.*

Consistency

- **Alternative Hypothesis**:

$$H_1 : \Lambda_0^+ \cup \Lambda_0^{\simeq} \neq \emptyset, \text{ where}$$

$$\Lambda_0^+ = \left\{ \lambda : \inf_{x \in \mathcal{X}} [G_Y(x) - G_\lambda(x)] > 0 \right\},$$

$$\begin{aligned} \Lambda_0^{\simeq} &= \left\{ \lambda : \inf_{x \in \mathcal{X}} [G_Y(x) - G_\lambda(x)] = 0 \right\} \cap \\ & \left\{ \lambda : \inf_{x \in B_\lambda^\varepsilon} [G_Y(x) - G_\lambda(x)] > 0 \text{ for some } \epsilon > 0 \right\}. \end{aligned}$$

- **Assumption 4.** *When* $\Lambda_0 = \Lambda_0^{\simeq}$,

$$\underline{\lim}_{T\to\infty} (T/u_T)^{1/2}\, \Delta_\lambda(\epsilon_T) > 0$$

for some $\lambda \in \Lambda_0^{\simeq}$, *where*

$$\Delta_\lambda(\epsilon) = \inf_{x \in B_\lambda^\epsilon} (G_Y(x) - G_\lambda(x))\,.$$

- **Theorem 3 (Consistency).** *Suppose that Assumptions 1-4 hold. Then, under the alternative hypothesis, we have*

$$P[W_T > s_{T,\hat{b}_T}(1-\alpha)] \to 1 \text{ as } T \to \infty.$$

Asymptotic Local Power

- **Local Alternatives:**

$$\mathbf{H}_a : G_Y(x) - G_{\lambda_T}(x) = \frac{\delta_{Y\lambda}(x)}{\sqrt{T}} \text{ for } \lambda_T \in \Lambda_{0T},$$

where

$$\Lambda_{0T} \quad = \quad \left\{\lambda + c/\sqrt{T} : \lambda \in \Lambda_0^{=},\ c \in \mathbb{R}^K\right\}$$

$$\delta_{Y\lambda}(\cdot) \quad : \quad \text{real function with } \inf_{x\in\mathcal{X}} [\delta_{Y\lambda}(x)] > 0.$$

- **Theorem 4.** *Suppose Assumptions 1 and 2 (with Λ_0 replaced by Λ_{0T}) hold. Then, under the sequence of local alternatives $\mathbf{H}_a$, we have*

$$W_T \Rightarrow \sup_{\lambda \in \Lambda_0^{=}} \inf_{x \in \mathcal{X}} \left[\widetilde{\nu}(\lambda, x) + \delta_{Y\lambda}(x) \right],$$

where $\widetilde{\nu}(\lambda, x)$ is defined as in Theorem 1.

Preliminary Simulation Results

- **Fundamental assets** $(K = 2)$:

$$X_t = \begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} \sim N\left(\begin{bmatrix} 0.698 \\ 0.833 \end{bmatrix}, \begin{bmatrix} 16.12 & 0 \\ 0 & 68.99 \end{bmatrix} \right)$$

- **Evaluated portfolios**:
  - ⋆ *Equally weighted portfolio (EP)* : Inefficient (H_1)
  - ⋆ *Tangency portfolio (TP)*: Efficient (H_0)

- **Parameters**:
  - ⋆ $T \in \{50, 100, 200, 500, 1000, 2000\}$.
  - ⋆ $k_T = 0.3\sqrt{\log(T)/T}$; $\epsilon_T = 2 \times k_T$.
  - ⋆ Number of Repetitions: $ns = 400$, $nb = 200$.
- **Figures 3 - 4:** *Median p-values for the SSDE test*
- **Figures 5 - 6:** *Median p-values for the TSDE test*

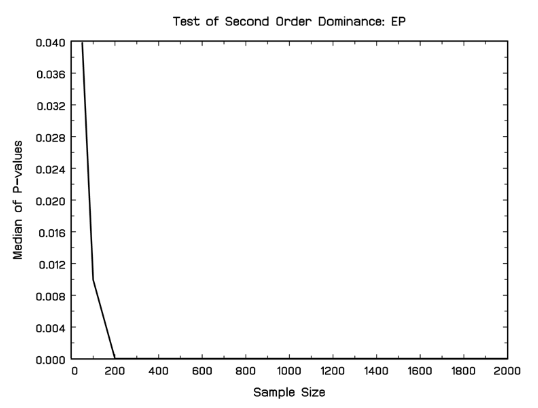


Figure 3. Alternative Hypothesis

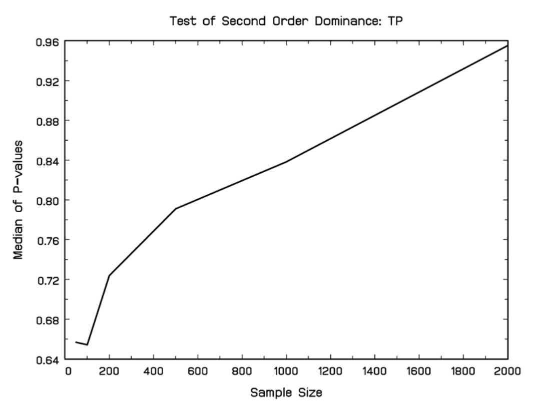


Figure 4. Null Hypothesis

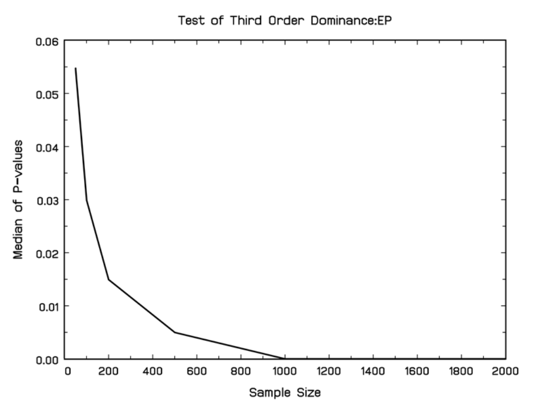


Figure 5. Alternative Hypothesis

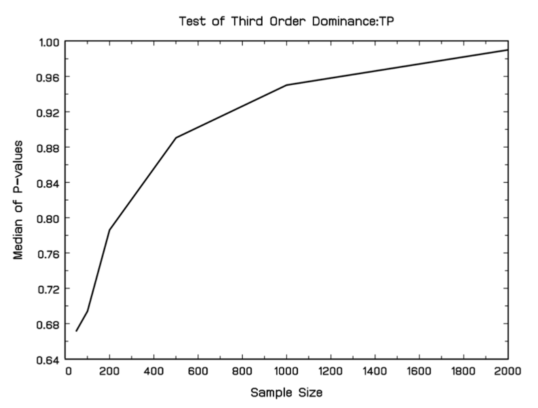


Figure 6. Null Hypothesis

Conclusion

- This paper develops a formal statistical test of stochastic dominance efficiency of a given portfolio with respect to an *infinite number of possible portfolios* from a given finite set of assets.
- This paper proposes a *nested linear programming algorithm* to deal with the nontrivial computational problem.
- The test is applicable to *general sampling schemes* and expected to be *more powerful* than some of the existing tests.
- Further investigation is needed on the choice of tuning parameters (k_T, ε_T), more simulation experiments when $K > 2$, and empirical applications, etc.