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A multiple variance ratio test using subsampling

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Abstract

This paper considers a test of the random walk hypothesis by comparing jointly the variance ratios at multiple observation intervals with unity. We suggest a subsampling procedure to approximate the asymptotic null distribution. Simulation results provide some favorable finite sample performance.

Keywords: Random walk hypothesis; Subsampling; Variance ratio test

JEL classification: C14; C52

1. Introduction

The variance ratio test, originally suggested by Lo and MacKinlay (1988), (hereafter LM), is commonly used in the finance and economics literature to test for the random walk hypothesis against stationary alternatives. Their test exploits the fact that the variance of random walk increments is linear in any and all observation intervals. However, their test focuses on testing one variance ratio at a time for a single observation interval, so it is essentially an *individual* hypothesis test. On the other hand, Chow and Denning (1993), (hereafter CD) propose that a proper test of random walk hypothesis should be based on a multiple comparison of a set of variance ratios with unity in order to have a correct overall size of the test, which requires a *joint* hypothesis test. Since the limit distribution of the CD's test statistic is very complicated, they suggest to use an upper bound on the asymptotic critical value. However, obvious weaknesses of the latter test are that it might be too conservative and hence may have significant size distortions and lower power compared to a test based on the exact asymptotic critical value. Another weakness is that their test is valid only when the

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sample autocorrelations of the random walk increments are asymptotically uncorrelated, which might not be true for some dependent time series.

The purpose of this paper is to develop a test that avoids the above weakness. Our test suggested below uses a subsampling procedure to directly approximate the asymptotic critical value. Also, contrary to a bootstrap procedure which imposes the stronger i.i.d. assumptions, it allows us to consider a *general heteroskedastic* time series for which sample autocorrelations of the increments may be *correlated*.

2. The test statistics and asymptotic null distributions

Let X_t denote a stochastic process satisfying the following recursive relation $X_t = \mu + X_{t-1} + \varepsilon_t$, $E(\varepsilon_t) = 0$ $\forall t \geq 1$,where μ is an arbitrary drift parameter. Under the random walk hypothesis, we have $E\varepsilon_t\varepsilon_{t-h} = 0$ $\forall h \neq 0$ $\forall t \geq 1$. Suppose that we obtain $N + 1$ observations $\{X_0, X_1, \ldots, X_N\}$, where $N = nq$ and both n and q are arbitrary integers greater than one. Let

$$M(q) \equiv \frac{\mathrm{Var}(X_{t+q} - X_t)/q}{\mathrm{Var}(X_{t+1} - X_t)} - 1. \tag{1}$$

Let $\{q_i\text{:}i = 1, \ldots, l\}$ denote a set of pre-specified aggregation intervals, where q_i is any integer greater than one with $q_i \neq q_j$ for $i \neq j$. We consider the following *joint* null and alternative hypotheses:

$$H_0\text{:}M(q_i) = 0 \text{ for all } i = 1, \ldots, l; \; H_1\text{:}M(q_i) \neq 0 \text{ for some } i = 1, \ldots, l. \tag{2}$$

Our test statistic is based on an estimator of $M(q_i), 1 \leq i \leq l$:

$$MV_N = \max_{1 \leq i \leq l} \left| \sqrt{N} \bar{M}(q_i) \right|, \text{ where} \tag{3}$$

$$\bar{M}(q) = \frac{\bar{\sigma}^2(q)}{\bar{\sigma}^2(1)} - 1, \; \bar{\sigma}^2(q) = \frac{1}{m} \sum_{t=q}^{N} \left(X_t - X_{t-q} - q\bar{X}\right)^2, \tag{4}$$

$m = q(N - q + 1)N - q)/N$, and $\bar{X} = (X_N - X_0)/N$. To derive the asymptotic distribution of MV_N, we assume:

Assumption 1. (a) $\{\varepsilon_t\text{:}t \geq 1\}$ is α-mixing sequence with mixing coefficient $\alpha(j)$ of size $r/(r-1)$, $r > 1$, such that for any $\tau \geq 0$, there exists some $\delta > 0$ for which $\sup_{t \geq 1} E|\varepsilon_t\varepsilon_{t-\tau}|^{2(r+\delta)} < \infty$. (b) $\lim_{N\to\infty} 1/N \sum_{t=1}^{N} E(\varepsilon_t^2) = \sigma_0^2$.

We now derive the limit null distribution. For $j = 1, \ldots, l$, let

$$\omega_j = \left(\frac{q_j - 1}{q_j} \; \cdots \; \frac{q_j - (q_j - 1)}{q_j} \right)'; \; \delta_{Nj} = \left(\frac{1}{\sqrt{N}} \sum_{t=1}^{N} \varepsilon_t \varepsilon_{t-1} \; \cdots \; \frac{1}{\sqrt{N}} \sum_{t=q_j-1}^{N} \varepsilon_t \varepsilon_{t-q_j+1} \right)'$$

and define

$$\Omega = 2 \cdot \mathrm{diag}\{\omega_1', \ldots, \omega_l'\}; \; \Delta = \lim_{N\to\infty} E(\delta_{N1}', \delta_{N2}', \ldots, \delta_{Nl}')'(\delta_{N1}', \delta_{N2}', \ldots, \delta_{Nl}').$$

Theorem 1. *Suppose Assumption 1 holds. Then, under the null hypothesis H_0, we have* $MV_N \xrightarrow{d} \max_{1\le i\le l}|Y_i|$ *as* $N\to\infty$, *where* $\{Y_1, \dots, Y_l\}$ *has a multivariate normal distribution with mean zero and covariance matrix* $\Phi=\sigma_0^{-4}\Omega\Delta\Omega'$.

Remark 1. In contrast to LM and CD, we derive Theorem 1 without imposing the restriction that the sample autocorrelations of ε_t are asymptotically uncorrelated, i.e., Assumption 4 of LM. Hence our test is robust to violations of this assumption.

Remark 2. The (asymptotic null) distribution of MV_N is that of a maximum of a multivariate normal random vector with an unknown covariance matrix, which is usually very complicated and can not be tabulated once and for all. We use a subsampling procedure to directly approximate the asymptotic null distribution, see Politis et al. (1999) for the main idea of subsampling [1].

3. Subsampling approximation

With some abuse of notation, we first write the test statistic MV_N as a function of the data $\{X_t: t=0, \dots, N\}$: i.e.,

$$MV_N=\sqrt{N}g_N(X_0,\dots,X_N),\ \text{where}\ g_N(X_0,\dots,X_N)=\max_{1\le i\le l}|\overline{M}(q_i)|.$$

Let $G_N(x)=P(\sqrt{N}g_N(X_0,\dots,X_N)\le x)$ denote the distribution function of MV_N. Let $g_{N,b,t}$ be equal to the statistic g_b evaluated at the subsample $\{X_t,\dots,X_{t+b-1}\}$ of size b: $g_{N,b,t}=g_b(X_t,\dots,X_{t+b-1})$ for $t=0,\dots,N-b+1$. We approximate the sampling distribution G_N of MV_N by $\hat{G}_{N,b}(x)=(N-b+2)^{-1}\sum_{t=0}^{N-b+1}1(\sqrt{b}g_{N,b,t}\le x)$. Let $g_{N,b}(1-\alpha)$ denote the $(1-\alpha)$-th sample quantile of $\hat{G}_{N,b}(\cdot)$, i.e., $g_{N,b}(1-\alpha)=\inf\{x:\hat{G}_{N,b}(x)\ge 1-\alpha\}$. We call it the *subsample critical value* of significance level α. Thus, we reject the null hypothesis at the significance level α if $MV_N>g_{N,b}(1-\alpha)$.

Let $g(1-\alpha)$ denote the $(1-\alpha)$-th quantile of the asymptotic null distribution of MV_N. We now justify the above subsampling procedure.

Theorem 2. *Suppose Assumption 1 holds. Assume $b/N\to 0$ and $b\to\infty$ as $N\to\infty$. Then, under the null hypothesis H_0, we have: as $N\to\infty$,*

$$\text{(a) } g_{N,b}(1-\alpha)\xrightarrow{p} g(1-\alpha);\ \text{(b) } P(MV_N>g_{N,b}(1-\alpha))\to\alpha.$$

The following theorem establishes the consistency of our test:

Theorem 3. *Suppose Assumption 1 holds. Assume $b/N\to 0$ and $b\to\infty$ as $N\to\infty$. Then, under the alternative hypothesis H_1, we have: as $N\to\infty$, $P(MV_N>g_{N,b}(1-\alpha))\to 1$.*

The assumptions on b are very weak and just require that the size of the subsamples goes to infinity

[1] Malliaropulos (1996) suggests a bootstrap procedure to approximate a limit distribution of a (single) variance ratio test. However, the bootstrap test imposes the stronger i.i.d. assumption for ε_t and hence is not applicable to general dependent series. Our subsampling approximation, however, works for dependent and heteroskedastic time series.

at a rate slower than the total sample size. In practice, the choice of b is important and rather difficult. It is similar to the problem of choosing optimal bandwidth in testing for parametric against nonparametric hypotheses. Some methodology for choosing b optimally would be desirable, but the main problem here is that usually b that is good for size distortion is not good for power and vice versa.[2]

4. Monte Carlo experiments

In this section, we report some numerical results that illustrate the finite sample performance of our test MV_N, comparing it to the tests Z_1^* and Z_2^* of CD. The designs we examined are similar to those of CD. The size of the tests is estimated under the random walk model $X_t = X_{t-1} + \varepsilon_t$ with (1) homoskedastic increments, i.e., ε_t is i.i.d. $N(0,1)$ (denoted HOMO) and (2) heteroskedastic increments, i.e., $\varepsilon_t = \sqrt{h_t}\eta_t$ where $h_t = 0.01 + \gamma_1 h_{t-1} + 0.2\varepsilon_{t-1}^2$ with $\gamma_1 = 0.45$ (GARCH1) and $\gamma_1 = 75$ (GARCH2). On the other hand, the power of the tests is estimated under (1) AR(1) model $X_t = 0.96X_{t-1} + \varepsilon_t$, where ε_t is i.i.d. $N(0,1)$ or GARCH(1,1) and (2) ARIMA(1,1,1) model $X_t = Y_t + Z_t$, where $Y_t = \theta Y_{t-1} + \varepsilon_t$, $\varepsilon_t \sim$ i.i.d. $N(0, 1)$, $Z_t = Z_{t-1} + \tau_t$, $\tau_t \sim$ i.i.d. $N(0,1/2)$, and $\theta \in (0.85, 0.96)$.

We considered $N \in (320, 640, 1280)$ and took $\{q_i : 1 \le i \le l\}$ to be $\{2, 4\}$ for $N = 320$, $\{2, 4, 8\}$ for $N = 640$, and $\{2, 4, 8, 16\}$ for $N = 1280$, respectively. An interesting practical question is how one should choose the subsample size given a specific sample size. To answer (partially) the question, we considered a total of six different subsamples (denoted $b1, \ldots, b6$) in a wide range for each sample size N: specifically, we took an equally spaced grid of subsample sizes on the range $2.5 \times N^{0.3} < b < 3.5 \times N^{0.6}$. Another interesting question is how our subsampling procedure compares to a bootstrap procedure which is based on 'full' samples. For this purpose, we have also considered a test based on the bootstrap critical value, i.e. MV_N^B.[3] We took the number of bootstrap replications $B = 500$. In each simulation, we did 1,000 replications.

Table 1 provides the size performance of the tests. We report the rejection probabilities with nominal size 0.05. The results show that the size performance of the test MV_N is reasonably good in all the designs we considered and it improves as the sample size increases. On the other hand, as expected, the bound tests of CD tend to under-reject the null hypothesis as the sample size increases. We also find that the bootstrap-based test MV_N^B has some size distortions especially in small/moderate samples for GARCH cases.

Tables 2–3 give the power performance of test against an AR(1) and ARIMA(1, 1, 1) alternatives. They show that our subsampling test is more powerful than Z_1^*, Z_2^* and MV_N^B in finite samples. The results were again robust to the subsample sizes.

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[2]Delgado et al. (2001) propose a method for selecting b to minimize size distortion in the context of hypothesis testing within the maximum score estimator, although no optimality properties of this method were proven.

[3]This bootstrap procedure imposes the i.i.d. assumption for ε_t (which is stronger than the null restriction) and hence would not be valid for dependent time series. However, unless the parametric model for ε_t series is known, it is hard to generate a generally dependent but uncorrelated time series via bootstrapping.

Table 1
Size performance

b		Z_1^*	Z_2^*	MV_N^B	MV_N					
					$b1$	$b2$	$b3$	$b4$	$b5$	$b6$
$N = 320$	HOMO	0.046	0.047	0.058	0.053	0.056	0.063	0.068	0.070	0.070
	GARCH1	0.064	0.044	0.077	0.064	0.060	0.065	0.070	0.073	0.074
	GARCH2	0.055	0.047	0.062	0.057	0.062	0.065	0.070	0.072	0.073
$N = 640$	HOMO	0.048	0.048	0.064	0.077	0.056	0.053	0.044	0.042	0.049
	GARCH1	0.070	0.052	0.074	0.090	0.052	0.052	0.055	0.049	0.053
	GARCH2	0.063	0.054	0.070	0.089	0.055	0.053	0.054	0.047	0.053
$N = 1280$	HOMO	0.041	0.039	0.055	0.062	0.059	0.056	0.049	0.045	0.054
	GARCH1	0.053	0.037	0.058	0.056	0.062	0.054	0.051	0.050	0.059
	GARCH2	0.048	0.041	0.052	0.061	0.062	0.057	0.048	0.048	0.059

Table 2
Power performance (ARIMA(1, 1, 1))

b		Z_1^*	Z_2^*	MV_N^B	MV_N					
					$b1$	$b2$	$b3$	$b4$	$b5$	$b6$
$\theta = 0.85$	$N = 320$	0.163	0.164	0.224	0.302	0.345	0.338	0.321	0.306	0.290
	$N = 640$	0.487	0.490	0.669	0.729	0.802	0.805	0.778	0.753	0.720
	$N = 1280$	0.964	0.965	0.996	0.997	0.999	1.00	1.00	0.999	0.999
$\theta = 0.96$	$N = 320$	0.047	0.041	0.052	0.098	0.097	0.084	0.076	0.090	0.092
	$N = 640$	0.057	0.059	0.083	0.158	0.165	0.175	0.163	0.161	0.153
	$N = 1280$	0.103	0.103	0.222	0.264	0.376	0.400	0.402	0.389	0.358

Table 3
Power performance (AR(1))

b		Z_1^*	Z_2^*	MV_N^B	MV_N					
					$b1$	$b2$	$b3$	$b4$	$b5$	$b6$
$N = 320$	HOMO	0.051	0.054	0.049	0.108	0.110	0.109	0.108	0.114	0.111
	GARCH1	0.068	0.048	0.082	0.114	0.110	0.116	0.118	0.112	0.108
	GARCH2	0.060	0.053	0.072	0.115	0.108	0.116	0.113	0.119	0.113
$N = 640$	HOMO	0.078	0.082	0.131	0.212	0.239	0.249	0.247	0.249	0.243
	GARCH1	0.101	0.072	0.148	0.215	0.240	0.247	0.241	0.250	0.228
	GARCH2	0.088	0.073	0.142	0.212	0.236	0.254	0.245	0.247	0.237
$N = 1280$	HOMO	0.286	0.293	0.496	0.534	0.602	0.664	0.692	0.685	0.672
	GARCH1	0.296	0.263	0.505	0.502	0.587	0.648	0.689	0.668	0.646
	GARCH2	0.299	0.268	0.512	0.522	0.591	0.655	0.686	0.670	0.659

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Appendix A

Proof of Theorem 1. Using Assumption 1, we can show that: for $i = 1, \ldots, l$,

$$\sqrt{N}M(q_i) = \frac{2}{\sigma_0^2 \sqrt{N}} \sum_{k=1}^{q_i - 1} \sum_{t=k+1}^{N} \left(\frac{q_i - k}{q_i} \right) \varepsilon_t \varepsilon_{t-k} + o_p(1).$$

Now the desired result of Theorem 1 holds by the CLT of Herrndorf (1984), Cramer–Wold device and continuous mapping theorem, see Whang and Kim (2002) for details. □

Proof of Theorems 2 and 3. Similar to the proof of Theorem 3.5.1(i) and (iii) of Politis et al. (1999), see Whang and Kim (2002) for details. □

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